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Spectral synthesis for Euler type operators

S. G. MERZLYAKOV

Let $A=(a_{i,j})_{i,j=1}^{k}$ be a square matrix. The operator

$$D_A = \sum_{i,j=1}^{k} a_{ij} z_j \frac{\partial}{\partial z_i}$$

is called an Euler type operator generated by the matrix A.

In this paper conditions are obtained on the matrix A and the domain $U \subset \mathbf{C}^k$, under which any closed invariant subspace W of the space $H(U)$ admits a spectral synthesis, i.e. eigen- and associated functions of the operator D_A which belong to W form a complete system on it.

§ I. Holomorphic dependence of functions

Let U be a domain in $\mathbf{C}^k$; $H(U)$ will denote the space of holomorphic functions in U with the uniform convergence topology on the compact subsets. For a compact set $K \subset \mathbf{C}^r$, $H(K)$ will denote the space of germs of the holomorphic functions in K with the inductive limit topology (see [5, p. 17]).

Let us consider the system $\varphi(t)=((\varphi_1(t), \ldots, \varphi_r(t))$ of complex functions on the ray $t \leqq 0$. Denote by $M(\varphi)$ the closure of the set $\{\varphi(t)\colon t \leqq 0\}$ in the space $\mathbf{C}^r$. We say that the system $\varphi(t)$ is holomorphically independent if there exists no function h different from zero and holomorphic in a neighbourhood of $M(\varphi)$ such that $h(\varphi(t))=0$ for all $t \leqq 0$.

We state the basic result of this section.

Theorem 1. *Consider a system of exponential monomials* $p_i(t)=t^{m_i}\exp \lambda_i t$, $i=1, \ldots, n$, *which are bounded on the ray* $t \leqq 0$. *Then there exists a holomorphically independent system* $\varphi(t)=(\varphi_1(t), \ldots, \varphi_r(t))$, $t \leqq 0$, *and a set of functions* $P_i \in H(M(\varphi))$, $i=1, \ldots, n$, *such that for each* i *we have* $p_i(t)=P_i(\varphi(t))$, $t \leqq 0$. *Moreover, if all* $m_i=0$, *then* $\varphi_j(t)=\exp \mu_j t$, $j=1, \ldots, r$, $\operatorname{Re} \mu_j>0$, *while in the opposite case* $\varphi_1(t)=t \exp \mu_1 t$, $\varphi_j(t)=\exp \mu_{j-1} t$, $j=2, \ldots, r+1$, $\operatorname{Re} \mu_1>0$, $\operatorname{Re} \mu_j \geqq 0$, $j=2, \ldots, r$.

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For the proof of this theorem we need some auxiliary results.

Lemma 1. *Assume the conditions of Theorem* 1 *are fulfilled. Then there exists a system of complex numbers* $\mu_1, \ldots, \mu_r \in \{\operatorname{Re}\mu \geqq 0\}$, *which is linearly independent over the field* Q, *such that the representations*

$$\lambda_i = \sum_{j=1}^{r} d_{ij}\mu_j, \quad d_{ij} \in \mathbf{Z}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, r,$$

hold and the conditions

a) $(\operatorname{Re}\lambda_i = 0,\ \operatorname{Re}\mu_j > 0) \Rightarrow d_{ij} = 0,$

b) $(\operatorname{Re}\lambda_i > 0,\ \operatorname{Re}\mu_j > 0) \Rightarrow d_{ij} \geqq m_i$

are fulfilled.

Proof. The boundedness of the functions $t^{m_i}\exp\lambda_i t$ on the ray $t \leqq 0$ implies that $\operatorname{Re}\lambda_i \geqq 0$, and $\operatorname{Re}\lambda_i > 0$ if $m_i > 0$. First we assume that $\operatorname{Re}\lambda_i > 0$ for all i and $\{\lambda_1, \ldots, \lambda_r\}$ is a maximal linearly independent subsystem of the system $\{\lambda_i : i = 1, \ldots, n\}$. In this case

$$\lambda_i = \sum_{j=1}^{r} b_{ij}\lambda_j, \quad i = 1, \ldots, n, \tag{1}$$

where $b_{ij} \in Q$. Let us introduce the matrices

$$B = (b_{ij})_{i=1;\, j=1}^{n;\, r}, \quad \Lambda(\varepsilon) = \big(\operatorname{Re}\lambda_i(1+\varepsilon\delta_{ij})\big)_{i,j=1}^{r},$$

where δ_{ij} denotes the Kronecker function.

Since $\operatorname{Re}\lambda_i > 0$, representation (1) implies that the entries of the matrix $B\Lambda(0)$ are positive, therefore the entries of the matrix $B\Lambda(\varepsilon)$ are also positive for sufficiently small ε. The determinant of the matrix $\Lambda(\varepsilon)$ is a nonzero polynomial of ε; consequently, for sufficiently small $\varepsilon \neq 0$ the matrix $\Lambda(\varepsilon)$ is invertible. For such an ε, let us set

$$\Lambda^{-1}(\varepsilon) = \big(\varkappa_{ij}(\varepsilon)\big)_{i,j=1}^{r}.$$

We have

$$\sum_{j=1}^{r} \varkappa_{ij}(\varepsilon)(1+\varepsilon\delta_{il})\operatorname{Re}\lambda_j = \delta_{il},$$

and consequently,

$$\sum_{l=1}^{r}\sum_{j=1}^{r} \varkappa_{ij}(\varepsilon)(1+\varepsilon\delta_{jl})\operatorname{Re}\lambda_j = \sum_{j=1}^{r}\sum_{l=1}^{r} \varkappa_{ij}(\varepsilon)(1+\varepsilon\delta_{il})\operatorname{Re}\lambda_j =$$

$$= (r+\varepsilon)\sum_{j=1}^{r} \varkappa_{ij}(\varepsilon)\operatorname{Re}\lambda_j = 1,$$

so that the entries of the matrix $\Lambda^{-1}(\varepsilon)\Lambda(0)$ are positive.

Thus, we have shown that, for sufficiently small $\varepsilon\neq 0$, the entries of the matrices $BA(\varepsilon)$, $A^{-1}(\varepsilon)A(0)$ are positive. Varying the entries of the matrix $A(\varepsilon)$ to sufficiently small values, we can find a matrix $C=(c_{ij})_{i,j=1}^r$ such that $c_{ij}\in Q$ and the entries of the matrices BC, $C^{-1}A(0)$ are positive. Put

$$BC=(d_{il})_{i=1;\,l=1}^{n\;\;\;r},\quad C^{-1}=(\varkappa_{ij})_{i,j=1}^r,\quad \mu_j=\sum_{l=1}^r \varkappa_{jl}\lambda_l,\quad j=1,\dots,r.$$

Then $\operatorname{Re}\mu_j>0$, $j=1,\dots,r$,

$$\lambda_i=\sum_{j=1}^r d_{ij}\mu_j,\quad i=1,\dots,n,\quad d_{ij}\in Q,\quad d_{ij}>0.$$

Replacing the system $\{\mu_j\colon j=1,\dots,r\}$ by the system $\{\mu_j/m\colon j=1,\dots,r\}$, where m is a sufficiently large integer, we obtain a system with the required properties.

Now we turn to the general case. Let us select subsystems $M_1\subset\{\lambda\in M\colon \operatorname{Re}\lambda=0\}$ and $M_2\subset\{\lambda\in M\colon \operatorname{Re}\lambda>0\}$ from the system $M=\{\lambda_1,\dots,\lambda_n\}$, such that M_1 forms a maximal linearly independent subsystem over Q of the system $\{\lambda\in M\colon \operatorname{Re}\lambda=0\}$, and $M_1\cup M_2$ forms that of M. Any $\lambda\in M$ admits the decomposition $\lambda=\lambda'+\lambda''$ where λ' and λ'' belong to the linear spans over Q of M_1 and M_2, respectively. From what we have proved above it follows that we can find a linearly independent system $\mu_1,\dots,\mu_r$, $\operatorname{Re}\mu_j>0, j=1,\dots,r,$ over Q such that if $\lambda_i\in M$ and $\operatorname{Re}\lambda_i>0$, then

$$\lambda_i''=\sum_{j=1}^r d_{ij}\mu_j,\quad d_{ij}\in\mathbf{Z},\quad d_{ij}\geqq m_i.$$

Dividing the elements of M_1 by a sufficiently large integer, we obtain a system $\mu_{r+1},\dots,\mu_k$ for which the numbers λ' are expanded with integral coefficients. It is easy to see that the system $\{\mu_1,\dots,\mu_k\}$ satisfies all requirements of Lemma 1.

Set

$$\varphi_j(t)=e^{\mu_j t},\quad j=1,\dots,r,\quad P_i(w)=\prod_{j=1}^r w_i^{d_{ij}},\quad w\in\mathbf{C}^r$$

if all $m_i=0$, and set

$$\varphi_1(t)=te^{\mu_1 t},\quad \varphi_j(t)=e^{\mu_{j-1}t},\quad j=2,\dots,r+1,\quad P_i(w)=w_1^{m_i}w_2^{c_{i1}-m_i}\prod_{j=3}^{r+1} w_j^{d_{ij-1}}$$

otherwise.

In order to complete the proof of Theorem 1, it suffices to show that the system $\varphi(t)$ is holomorphically independent. This follows from the following.

Lemma 2. *Let* $\alpha_n\in\mathbf{Z}$, $\alpha_n\geqq 0$, $\beta_n\in\mathbf{C}$, $n=1,\dots,$ *let the pairs* (α_n,β_n) *be different for different n-s, let the set* $\{\operatorname{Re}\beta_n\leqq b\}$ *be finite for all b, and finally suppose that there exists* $c>0$ *such that* $\alpha_n\leqq c\operatorname{Re}\beta_n$ *for all n. If the sequence* $a_n\in\mathbf{C}$, $n\geqq 1$ *is such*

that, for $t \leqq t_0 \leqq 0$, *the series*

$$\sum_{n=1}^{\infty} a_n t^{\alpha_n} e^{\beta_n t}$$

converges absolutely to 0, *then* $a_n = 0$, $n = 1, \ldots$.

Proof. Assume that the statement of the lemma fails, and set

$$\beta = \min\{\operatorname{Re}\beta_n\colon\ a_n \neq 0\}, \quad \alpha = \max\{\alpha_n\colon\ \operatorname{Re}\beta_n = \beta,\ a_n \neq 0\}.$$

Since the set $\{\operatorname{Re}\beta_n\}$ is discrete, there exists $\delta > 0$ such that $\operatorname{Re}\beta_n \geqq \beta + \delta$ if $\operatorname{Re}\beta_n > \beta$. For $c_1 = (c\beta - \alpha)\delta^{-1} + c$, the inequality

$$(\alpha_n - \alpha) \leqq c_1(\operatorname{Re}\beta_n - \beta)$$

holds if $a_n \neq 0$.

Let us fix an $\varepsilon > 0$. There exists a number $t_1 \leqq t_0$ such that, for $t \leqq t_1$, $\operatorname{Re}\beta_n = \beta$, $\alpha_n < \alpha$, we have

$$(\alpha_n - \alpha)(\ln|t| - \ln|t_0|) \leqq \ln\varepsilon,$$

and for $\operatorname{Re}\beta_n > \beta$ we have

$$(\alpha_n - \alpha)(\ln|t| - \ln|t_0|) + (\operatorname{Re}\beta_n - \beta)(t - t_0) \leqq c_1(\operatorname{Re}\beta_n - \beta)(\ln|t| - \ln|t_0|) +$$
$$+ (\operatorname{Re}\beta_n - \beta)(t - t_0) = (\operatorname{Re}\beta_n - \beta)[c_1(\ln|t| - \ln|t_0|) + (t - t_0)] \leqq \ln\varepsilon.$$

For $t \leqq t_1$ we obtain

$$\Big|t^{-\alpha} e^{-\beta t} \sum_{(\alpha_n,\ \operatorname{Re}\beta_n) \neq (\alpha, \beta)} a_n t^{\alpha_n} e^{\beta_n t}\Big| \leqq \varepsilon |t_0^{-\alpha} e^{-\beta t_0}| \sum_{n=1}^{\infty} |a_n t_0^{\alpha_n} e^{\beta_n t_0}|.$$

So, we have shown that

$$\lim_{t \to -\infty} \Big[t^{-\alpha} e^{-\beta t} \sum_{(\alpha_n,\ \operatorname{Re}\beta_n) = (\alpha, \beta)} a_n t^{\alpha_n} e^{\beta_n t}\Big] = 0. \tag{2}$$

The function inside the brackets is almost periodic on the real axis and according to (2), $a_n = 0$ follows for $(\alpha_n, \operatorname{Re}\beta_n) \neq (\alpha, \beta)$ (see [7, pp. 239—250]). This contradicts our assumption.

Corollary. *Let* $\mu_1, \ldots, \mu_p \in \mathbf{C}$, $\operatorname{Re}\mu_1 > 0$, $\operatorname{Re}\mu_j \geqq 0$, $j = 2, \ldots, p$, *and let the system* $\mu_1, \ldots, \mu_p$ *be linearly independent over* Q. *Then the system of functions*

$$\varphi_1(t) = t \exp \mu_1 t, \quad \varphi_j(t) = \exp \mu_{j-1} t, \quad j = 2, \ldots, p+1,$$

is holomorphically independent.

Proof. By virtue of Kronecker's theorem (see [2, p. 314]), the set $M(\varphi)$ contains a subset $\{(w_1, \ldots, w_{p+1})$ such that $|w_i| = 0$ if $\operatorname{Re}\mu_i > 0$, and $|w_i| = 1$ if $\operatorname{Re}\mu_i = 0$, $i = 1, \ldots, p+1\}$. So, the functions $h \in H(M(\varphi))$ can be expanded into a Laurent series

$$h(w) = \sum a_\alpha w^\alpha$$

in a neighbourhood of this set. Let the function h be equal to zero on $M(\varphi)$. Then there exists $t_0 \leqq 0$ such that, for $t \leqq t_0$, the series

$$\sum a_\alpha t^{\alpha_1} e^{\alpha\mu t}, \quad \alpha\mu = \alpha_1\mu_1 + \sum_{j=1}^{p} \alpha_{j+1}\mu_j,$$

converges absolutely to zero. We shall show that this series satisfies the conditions of Lemma 2.

The set $\{\operatorname{Re}\alpha\mu \leqq b\}$ is finite, since

$$\operatorname{Re}\alpha\mu \geqq \Big(\sum_{j=1}^{p+1} \alpha_j\Big) \min_{\operatorname{Re}\mu_j \neq 0} \operatorname{Re}\mu_j.$$

The pairs $(\alpha_1, \alpha\mu)$ are different by virtue of the linear independence of the system $\{\mu_2, ..., \mu_p\}$, and the number $c = (\operatorname{Re}\mu_1)^{-1}$ satisfies the inequality

$$\alpha_1 \leqq c \operatorname{Re}\alpha\mu.$$

Thus, by Lemma 2, $a_\alpha = 0$ and $h \equiv 0$.

Now, we prove the statement which allows us to construct holomorphically dependent systems of functions.

Lemma 3. *Let $D \subset \mathbf{C}^n$ be a domain of holomorphy, $U \subset \mathbf{C}^k$ be an arbitrary domain, the holomorphic mapping $F\colon U \to D$ be proper, and $m > k$. Then there exists a function $h \in H(D)$ such that $h \not\equiv 0$ and $h \circ F \equiv 0$ on U.*

Proof. Since $m > k$, there exists a point $w \in D$ such that $w \notin F(U)$. By a theorem of REMMERT [3, p. 55], the set $F(U)$ is analytic in the domain D. Therefore, the set $\{w\} \cup F(U)$ is also analytic. Every function, which is zero on the set $F(U)$ and equals 1 at the point w, is analytic on the set $\{w\} \cup F(U)$. By a theorem of CARTAN [4, p. 313] there exists a function $h \in H(D)$, $h(w) = I$, $h \circ F \equiv 0$ on U. The lemma is proved.

Example. The system (te^t, e^{-at}), $a > 0$, is holomorphically dependent. Indeed, the mapping $F(w) = (we^w, e^{-aw})$ from $\mathbf{C}$ to $\mathbf{C}^2$ is proper: if

$$w_n \to \infty, \quad w_n \exp w_n \to \alpha, \quad \exp(-aw_n) \to \beta,$$

where $\alpha, \beta \in \mathbf{C}$, then

$$(\exp w_n \to 0) \Rightarrow (\operatorname{Re} w_n \to -\infty) \Rightarrow (e^{-aw_n} \to \infty),$$

which is a contradiction.

From Lemma 3 we obtain that there exists a function $h \in H(\mathbf{C}^2)$, $h \not\equiv 0$, such that

$$h(we^w, e^{-aw}) = 0, \quad w \in \mathbf{C}.$$

§ 2. A Mittag-Leffler type theorem for invariant subspaces

In this section we show that any closed D_A-invariant subspace of the space $H(U)$ admits a spectral synthesis, provided that for any point $z \in U$ the closure of the set $\{(\exp tA)z : t \leqq 0\}$ is compact and lies in the domain U.

Definition. We call a domain $U \subset \mathbf{C}^k$ A-star like if for every $z \in U$ the set $\{(\exp tA)z : t \leqq 0\}$ lies in the domain U. We call U *strongly* A-star like if the closures of these sets also lie in U.

The operator D_A admits the following representation.

Lemma 4. *If a function f belongs to $H(U)$, then*

$$(D_A^m f)(z) = \frac{\partial^m}{\partial t^m}\bigg|_{t=0} f(e^{tA}z), \quad m \geqq 0.$$

Proof. By definition,

$$(D_A f)(z) = \left(\frac{\partial f}{\partial z_i}\right)_{i=1}^k Az,$$

therefore

$$\frac{\partial}{\partial t} f(e^{tA}z) = \frac{\partial}{\partial \tau}\bigg|_{\tau=0} f(e^{(t+\tau)A}z) = \left(\frac{\partial f}{\partial z_i}\right)_{i=1}^k e^{tA}Az = (D_A f)(e^{tA}z).$$

Arguing further by induction, the lemma will be proved.

Corollary. *Let $U \subset \mathbf{C}^k$ be an A-star like domain and $W \subset H(U)$ be a closed D_A-invariant subspace. Then*

$$f \in W \Rightarrow f(e^{tA}z) \in W, \quad t \leqq 0.$$

Proof. Let us consider an arbitrary linear continuous functional S on the space $H(U)$ which annihilates the subspace W. There exists a measure ν with compact support $K \subset U$ such that for $g \in H(U)$

$$\langle S, g \rangle = \int_K g(z)\, d\nu(z).$$

The function

$$\psi(h) = \int_K f(e^{hA}z)\, d\nu(z)$$

is holomorphic in a neighbourhood of the ray $t \leqq 0$. Let us consider the derivatives of this function at zero:

$$\psi^{(m)}(0) = \int_K \frac{\partial^m}{\partial t^m}\bigg|_{t=0} f(e^{tA}z)\, d\nu(z) = \int_K (D_A^m f)(z)\, d\nu(z) = \langle S, D_A^m f \rangle = 0.$$

Thus, for $t \leqq 0$,

$$\langle S, f(e^{tA}z)\rangle = 0.$$

By the Hahn—Banach theorem, $f(e^{tA}z) \in W$. The Corollary is proved.

Let us consider an arbitrary system $\varphi(t) = (\varphi_1(t), \ldots, \varphi_p(t))$ of bounded continuous functions on $(-\infty, 0]$ and a quadratic matrix $E(w)$ of order k, whose entries are holomorphic in a domain $\sigma \supset M(\varphi)$ and such that $E(\varphi(t)) = \exp tA$, $t \leqq 0$.

For a strongly A-star like domain U and a compact set $K \subset U$, put

$$(U, K) = \{w \in \sigma \colon E(w)K \subset U\}.$$

This set is open and contains the compact set $M(\varphi)$. Indeed, for any point $w \in M(\varphi)$ there exists a sequence $\{t_n\}$, $t_n \leqq 0$ such that $\varphi(t_n) \to w$. Therefore, for $z \in U$ we have

$$E(w)z = \lim_{n\to\infty} E(\varphi(t_n))z = \lim_{n\to\infty} e^{t_n A}z \in U$$

because the domain U is strongly A-star like.

To an arbitrary functional $T \in H^*(M(\varphi))$ there corresponds an operator $\tilde{T}$ acting on the space $H(U)$ such that

$$(\tilde{T}f)(z) = \langle T, f(E(w)z)\rangle.$$

If $K \subset U$ is compact, then the function $f(E(w)z)$ is holomorphic with respect to the variable w on the set (U, K) and with respect to z in the interior of the compact set K. Thus, the function $\tilde{T}f$ is holomorphic on U.

Assume that the system of functions $T_n \in H^*(M(\varphi))$, $n = 1, 2, \ldots$, has the uniqueness property

$$(h \in H(M(\varphi)),\ \langle T_n, h\rangle = 0,\ \forall n \geqq 1) \Rightarrow h \equiv 0.$$

In this case the following result is valid.

Lemma 5. *There exists a system consisting of the numbers $e_{nm} \in \mathbf{C}$, $n \geqq 1$, $m = 1, \ldots, n$, such that for any strongly A-star like domain $U \subset \mathbf{C}^k$ and function $f \in H(U)$ we have*

$$f(z) = \lim_{n\to\infty} \sum_{m=1}^{n} e_{nm}(\tilde{T}f)(z).$$

The convergence is meant in the topology of the space $H(U)$.

Proof. The space of the germs of holomorphic functions on the compact set $H(M(\varphi))$ is an $(\mathscr{L}N^*)$ space (see [6], [5, p. 18]); moreover, its dual $H^*(M(\varphi))$ is an (M^*) space, and consequently, it is metrizable and reflexive.

The system $\{T_n\}$ is complete in the space $H^*(M(\varphi))$ by virtue of the uniqueness property and the Hahn—Banach theorem. Therefore, for the functional $T_0 \in$

$\in H^*(M(\varphi))$, $\langle T_0, h\rangle = h(\varphi(0))$ and there exists a sequence $e_{nm} \in \mathbf{C}$ such that

$$\lim_{n\to\infty} \sum_{m=1}^{n} e_{nm} T_m = T_0$$

in the topology of the space $H^*(M(\varphi))$, i.e. uniformly on any bounded set of the space $H(M(\varphi))$. If $K \subset U$ is compact, then the set of functions $\{f(E(w)z)\colon z \in K\}$ of the variable w is bounded on the space $H(M(\varphi))$, so that

$$\lim_{n\to\infty} \sum_{m=1}^{n} e_{nm} \langle T_n, f(E(w)z)\rangle = f(E(\varphi(0)z)) = f(z)$$

uniformly on K. The lemma is proved.

Corollary. *Let $\lambda_1, \ldots, \lambda_k \in \mathbf{C}$, $\operatorname{Re}\lambda_i > 0$, $i = 1, \ldots, k$. Then there exists a sequence $e_{n\alpha}$ (α -multiindex of order k) such that for any domain $U \subset \mathbf{C}^k$ with the properties*

a) $0 \in U$,

b) $z \in U$, $t \leqq 0 \Rightarrow (e^{\lambda_1 t} z_1, \ldots, e^{\lambda_k t} z_k) \in U$,

and for any function $f \in H(U)$ the representation

$$f(z) = \lim_{n\to\infty} \sum_{|\alpha| \leqq n} e_{n\alpha} f^{(\alpha)}(0) z^\alpha$$

holds in the topology of the space $H(U)$.

Proof. Set $\varphi(t) = (\exp \lambda_1 t, \ldots, \exp \lambda_k t)$, $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$, $E(w) = \operatorname{diag}(w_1, \ldots, w_k)$, $\langle T_\alpha, h\rangle = h^{(\alpha)}(0)$. Then the domain U with the properties a) and b) is strongly A-star like, $(\tilde{T}_\alpha f)(z) = f^{(\alpha)}(0) z^\alpha$, and the statement follows from Lemma 5.

Remark. For $k = 1$, the converse result is also valid (see [1, p. 38]).

Lemma 6. *Let the system $\varphi(t)$ be holomorphically independent and the domain $U \subset \mathbf{C}^k$ be strongly A-star like. Then for any functional $T \in H^*(M(\varphi))$ and closed D_A -invariant subspace $W \in H(U)$, the operator $\tilde{T}$ maps W into W.*

Proof. Let f be an element of W and suppose that the functional $S \in H^*(U)$ annihilates the space W. There exists a measure ν with compact support $K \subset U$ such that, for $g \in H(U)$,

$$\langle S, g\rangle = \int_K g(z)\, d\nu(z).$$

The function

$$h(w) = \int_K f(E(w)z)\, d\nu(z)$$

is holomorphic on the set (U, K) and, according to Lemma 4, the equality

$$h(\varphi(t)) = 0, \quad t \leqq 0,$$

is fulfilled. Therefore, $h(w)=0$ in a connected component containing the set $M(\varphi)$. Using Fubini's theorem, we obtain

$$\langle S, Tf(E(w)z)\rangle = 0.$$

By the Hahn—Banach theorem, $\tilde{T}f \in W$. The lemma is proved.

Assume that the entries of the matrix $\exp tA$ are bounded for $t \leqq 0$, that is, they are linear combinations of the functions $t^{m_i} \exp \lambda_i t$, $\operatorname{Re} \lambda_i \geqq 0$, and if $\operatorname{Re} \lambda_i \geqq 0$, then $m_i=0$, $i=1, \dots, k$. For definiteness we will assume that not all the m_i are zero. By Theorem 1, there exists a holomorphically independent system of functions $\varphi(t)=(t \exp \mu_1 t, \exp \mu_1 t, \exp \mu_2 t, \dots, \exp \mu_r t)$ and a quadratic matrix $E(w)$ of order k with entries in the set $H(M(\varphi))$ such that $\operatorname{Re} \mu_1 > 0, \dots, \operatorname{Re} \mu_p > 0$, $\operatorname{Re} \mu_{p+1}=0, \dots$ $\dots, \operatorname{Re} \mu_r = 0$ and $E(\varphi(t)) = \exp tA$.

Since the system $\mu_{p+1}, \dots, \mu_r$ is linearly independent over Q, it follows from Kronecker's theorem that $M(\varphi)$ contains the set

$$\prod_{j=0}^{p} \{0\} \times \prod_{j=p+1}^{r} \{|w_j| = 1\}. \tag{3}$$

An arbitrary function $h \in H(M(\varphi))$ can be expanded into a Laurent series

$$h(w) = \sum \langle T_\alpha, h\rangle w^\alpha$$

in a neighbourhood of the set (3), where $\alpha = (\alpha_1, \dots, \alpha_r)$ is a multiindex, $\alpha_1, \dots, \alpha_p$ are nonnegative integers, and $\alpha_{p+1}, \dots, \alpha_r$ are integers. The functionals T_α belong to the space $H^*(M(\varphi))$ and form a complete system in it, because $M(\varphi)$ is a connected compact set.

For the functionals T_α the following result is valid.

Lemma 7. *Let $U \subset \mathbf{C}^k$ be a strongly A-star like domain and $f \in H(U)$. Then, for any multiindex α, the function $\tilde{T}_\alpha f$ is a root function of the operator D_A.*

Proof. Let us consider an arbitrary point z_0 in U and let K be the closure of the set $\{(\exp tA)z_0 \colon t \leqq 0\}$. By virtue of the conditions on the matrix A and the domain U, K is compact and lies in U.

For $z \in K$, the function $f(E(w)z)$ can be expanded into a Laurent series with respect to the variable w in a neighbourhood G of the set (3) as follows

$$f(E(w)z) = \sum (\tilde{T}_\beta f)(z) w^\beta, \quad z \in K, \quad w \in G.$$

There exists $\tau_0 \leqq 0$ such that, for $\tau \leqq \tau_0$, the point $\varphi(t)$ belongs to the set G, and consequently,

$$f(e^{\tau A} z) = \sum (\tilde{T}_\beta f)(z) \tau^{\beta_1} e^{\beta \mu \tau}, \quad z \in K, \quad \tau \leqq \tau_0,$$

where

$$\beta\mu = (\beta_1+\beta_2)\mu_1+\beta_3\mu_2+\ldots+\beta_{r+1}\mu_r.$$

Since for $t \leqq 0$ we have $(\exp tA)z_0 \in K$, it follows that on the one hand,

$$f(e^{(\tau+t)A}z_0) = f(e^{\tau A}e^{tA}z_0) = \sum (\tilde{T}_\beta f)(e^{tA}z_0)\tau^{\beta_1}e^{\beta\mu\tau}$$

and, on the other hand,

$$f(e^{(\tau+t)A}z_0) = \sum (\tilde{T}_\beta f)(z_0)(\tau+t)^{\beta_1}e^{\beta\mu(\tau+t)}.$$

The absolute convergence of the last series implies the absolute convergence of the series

$$\sum (\tilde{T}_\beta f)(z_0)\binom{\beta_1}{S}\tau^{\beta_1-S}t^S e^{\beta\mu(\tau+t)}.$$

Comparing the coefficients of $\tau^{\alpha_1}\exp\alpha\mu\tau$ in the two expansions of the function $f((\exp(\tau+t)A)t_0)$ and using Lemma 2 yields

$$(\tilde{T}_\alpha f)(e^{tA}z_0) = \sum_{S=0}^{\alpha_2}\binom{\alpha_1+S}{S}(\tilde{T}_{(\alpha_1+S,\alpha_2-S,\alpha_3,\ldots,\alpha_{r+1})}f)(z_0)t^S e^{\alpha\mu t}.$$

Hence, according to Lemma 4, we have

$$D_A\tilde{T}_\alpha = (\alpha_1+1)\tilde{T}_{(\alpha_1+1,\alpha_2-1,\alpha_3,\ldots,\alpha_{r+1})}+\alpha\mu\tilde{T}_2 \quad \text{if} \quad \alpha_2>0,$$

and

$$D_A\tilde{T}_\alpha = \alpha\mu\tilde{T}_\alpha \quad \text{if} \quad \alpha_2=0.$$

Thus,

$$(D_A-\alpha\mu I)^{\alpha_2+1}\tilde{T}_\alpha = 0,$$

where $I\colon H(U)\to H(U)$ is the identity operator. The lemma is proved.

Summing up what has been proved in the last three lemmas we can deduce the following.

Theorem 2. *Let A be a quadratic matrix of order k such that the entries of the matrix $\exp tA$ are bounded on the ray $t \leqq 0$. Then there exists a linear continuous operator $\mathcal{L}_\alpha$ on the space of the functions holomorphic on the strongly A-star like domains of $\mathbf{C}^k$ and there exists a sequence of numbers $e_{n\alpha}$ such that, for any strongly A-star like domain $U\subset\mathbf{C}^k$ and closed D_A-invariant subspace W of the space $H(U)$, the representation*

$$f\in W \Rightarrow f(z) = \lim_{n\to\infty}\sum_{|\alpha|\leqq n} e_{n\alpha}\mathcal{L}_\alpha f$$

is valid in the topology of $H(U)$. The function $\mathcal{L}_\alpha f$ also belongs to W and is a root function of D_A.

§ 3. Subspaces without spectral synthesis

In this section we present examples demonstrating the exactness of Theorem 2.

First, we shall show that, without the requirement of star-likeness of the domain U, possibly there exists no synthesis even for $k=1$.

Let the domain be defined by $U=\mathbf{C}\setminus(\{\operatorname{Im} z=0,\ \operatorname{Re} z\geqq 1\}\cup\{\operatorname{Im} z\geqq 0,\ \operatorname{Re} z=1\})$ and the space by

$$W=\{f\in H(\mathbf{C}\setminus\{1,-1\})\colon f(z)\to 0 \quad \text{as} \quad z\to\infty, \quad \text{and} \quad f(-z)=f(z)\}.$$

The space W is closed in the space $H(U)$ since the functions in W are even and the sections of U are nonsymmetric. If $f\in W$, then $zf'(z)\in W$.

In W there is no eigenfunction for the operator $z\partial/\partial z$ because if $g\in W$ and $zg'(z)=\lambda g(z)$ for any $\lambda\in\mathbf{C}$, then the expansion of g in a neighbourhood of zero implies that g is an entire function, and consequently, it is identically zero.

We shall give an example which shows the importance of the condition of strongly star-likeness.

Let A be a diagonal matrix of second order with eigenvalues 1 and i, U be the domain $(\mathbf{C}\setminus\{0\})^2$. If $z\in U$, then $(\exp tA)z\in U$ for any $t\in\mathbf{C}$.

Let us consider the space

$$W=\{g\in H(U)\colon g(e^t, e^{it})=0,\ t\in\mathbf{C}\}.$$

This is a closed subspace of $H(U)$ and invariant for the operator D_A. The mapping $h\colon \mathbf{C}\to U$, $h(t)=(e^t, e^{it})$ is proper, therefore Lemma 3 implies that the space W is nontrivial. However, there is no eigenfunction for the operator D_A in W. Indeed, let $D_A g=\lambda g$ be valid for some $g\in W$ and $\lambda\in\mathbf{C}$. Consider the expansion of g into a Laurent series

$$g(z)=\sum a_\alpha z^\alpha$$

and apply the operator D_A. We obtain that $(\alpha_1+i\alpha_2-\lambda)a_\alpha=0$ for all α, therefore $g(z)=a_{\alpha 0}z^{\alpha 0}$ with certain integers α_1^0 and α_2^0, but we have $a_{\alpha 0}=0$ since $g\in W$.

Theorem 2 implies that if for a quadratic matrix A of order k there exists $\gamma\in\mathbf{C}$, $\gamma\neq 0$, such that the entries of the set of matrices $\{\exp t\gamma A\colon t\leqq 0\}$ are uniformly bounded, then any closed D_A-invariant subspace of $H(\mathbf{C}^k)$ admits a spectral synthesis. The converse result is also valid.

Theorem 3. *Let A be a quadratic matrix of order k such that for any $\gamma\in\mathbf{C}$, $\gamma\neq 0$, the entries of the set of matrices $\{\exp t\gamma A\colon t\leqq 0\}$ are not uniformly bounded. Then there exists a closed D_A-invariant nontrivial subspace of $H(\mathbf{C}^k)$ without any eigenfunction of D_A.*

Proof. We may assume that A has a Jordan form. First we examine the following cases:

1) $k=2$, $A=\begin{pmatrix}0&1\\0&0\end{pmatrix}$. We take
$$W=\{g(z_2)e^{z_1}\colon\ g\in H(\mathbf{C})\}.$$
For a function f in W we have $(D_Af)(z)=z_2f(z)$ and this implies that there exists no eigenfunction in W.

$$2)\ k=3,\quad A=\begin{pmatrix}\lambda&1&0\\0&\lambda&0\\0&0&-b\lambda\end{pmatrix},\quad b\in\mathbf{R},\quad b>0,\quad \lambda\neq 0.$$

If b is an irrational number, then set
$$W=\{f\in H(\mathbf{C}^3)\colon\ f(te^{\lambda t},e^{\lambda t},e^{-b\lambda t})=0,\ t\in\mathbf{C}\}.$$
The mapping
$$h\colon\ \mathbf{C}\to\mathbf{C}^3,\quad h(w)=(we^{\lambda w},e^{\lambda w},e^{-b\lambda w})$$
is proper. Therefore, by Lemma 3, W is a nontrivial space.

We shall find the eigenfunctions of D_A. Let $g\in H(\mathbf{C}^3)$,
$$g(z)=\sum_{|\alpha|\geq 0}a_\alpha z^\alpha,\quad D_Ag=\mu g.$$
The Taylor coefficients satisfy the relations
$$(\mu-\lambda\alpha_1+b\lambda\alpha_3)a_{\alpha_1 0\alpha_3}=0,$$
$$(\mu-\lambda\alpha_1-\lambda\alpha_2+b\lambda\alpha_3)a_{\alpha_1\alpha_2\alpha_3}-(\alpha_1-1)a_{\alpha_1+1\,\alpha_2-1\,\alpha_3}=0.$$
Hence, we obtain
$$(\mu-\lambda\alpha_1-\lambda\alpha_2+b\lambda\alpha_3)^{\alpha_2}a_{\alpha_1\alpha_2\alpha_3}=\frac{(\alpha_1+\alpha_2)!}{\alpha_1!}a_{\alpha_1+\alpha_2 0\alpha_3},$$
$$\alpha_1!\,a_{\alpha_1\alpha_2\alpha_3}=(\mu-\lambda\alpha_1-\lambda\alpha_2+b\lambda\alpha_3)^{\alpha_1}a_{0\,\alpha_1+\alpha_2\,\alpha_3}.$$
If $\lambda\alpha_1+\lambda\alpha_2-b\lambda\alpha_3\neq\mu$ or $\lambda\alpha_1+\lambda\alpha_2-b\lambda\alpha_3=\mu$ and $\alpha_1\neq 0$, then it follows from the above mentioned equalities that $\alpha_\alpha=0$. Since b is an irrational number, there exists at most one pair (α_2,α_3) such that $\lambda\alpha_2-b\lambda\alpha_3=\mu$. Thus,
$$g(z)=cz_2^{\alpha_2}z_3^{\alpha_3}.$$
The function g belongs to W only in the case if $c=0$.

Let now b be a rational number, $b=p/q$ where p, q are positive integers. If we set
$$W=\{g(z_2^pz_3^q)e^{z_1z_2^{p-1}z_3^q}\colon\ g\in H(\mathbf{C})\},$$
then using Lemma 4, for $f\in W$ we obtain
$$(D_Af)(z)=z_2^pz_3^qf(z).$$
This implies that W is D_A-invariant and does not contain eigenfunctions for D_A.

3) $k=3$ and the origin is an inner point of the convex hull of the set of eigenvalues $\{\lambda_1, \lambda_2, \lambda_3\}$ for the matrix A. If the system $\lambda_1, \lambda_2, \lambda_3$ is linearly independent over Q, then set

$$W=\{f\in H(\mathbf{C}^3)\colon f(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})=0\}.$$

Arguing as in Section 2) it can be proved that this space satisfies the conditions of the theorem.

Let now the system $\lambda_1, \lambda_2, \lambda_3$ be linearly dependent, $k_1\lambda_1+k_2\lambda_2+k_3\lambda_3=0$, where k_1, k_2, k_3 are integers. From the assumption on this system it follows that k_1, k_2, k_3 differ from 0, are of the same sign, and we may assume that each of them is positive.

For W put

$$W=\left\{f\in H(\mathbf{C}^3)\colon f(e^{\lambda_1 t}, e^{\lambda_2 t}, e^{\lambda_3 t})=0,\ \frac{\partial^3 f}{\partial z_1 \partial z_2 \partial z_3}\equiv 0\right\}.$$

It is easy to see that this is a closed D_A-invariant space. We shall show that it contains nonzero functions. The mapping

$$h\colon \mathbf{C}\to(\mathbf{C}\setminus\{0\})\times(\mathbf{C}\setminus\{0\}),\quad h(w)=(e^{k_1\lambda_1 w}, e^{k_2\lambda_2 w}),$$

is proper because $\lambda_1/\lambda_2\notin\mathbf{R}$. Therefore, there exists a nonzero function $\varphi\in H((\mathbf{C}\setminus\{0\})\times(\mathbf{C}\setminus\{0\}))$ for which

$$\varphi(\exp k_1\lambda_1 w, \exp k_2\lambda_2 w)=0.$$

Let

$$\varphi(v_1, v_2)=\sum a_{\alpha_1\alpha_2} v_1^{\alpha_1} v_2^{\alpha_2},$$

where α_1, α_2 are integers. Let us introduce the following entire functions of three variables:

$$f_1(z)=\sum_{\alpha_1,\alpha_2\geqq 0} a_{\alpha_1\alpha_2} z_1^{k_1\alpha_1} z_2^{k_2\alpha_2},\quad f_2(z)=\sum_{\alpha_1\geqq\alpha_2,\ \alpha_2<0} a_{\alpha_1\alpha_2} z_1^{k_1(\alpha_1-\alpha_2)} z_3^{-k_3\alpha_2},$$

$$f_3(z)=\sum_{\alpha_1\leqq\alpha_2,\ \alpha_1<0} a_{\alpha_1\alpha_2} z_2^{k_2(\alpha_2-\alpha_1)} z_3^{-k_3\alpha_1}.$$

The function $f=f_1+f_2+f_3$ differs from zero because at least one of its Taylor coefficients is nonzero and $f\in W$.

Let $g\in W$, $D_A g=\mu g$. Expanding the function into a Taylor series and applying the operator D_A, we obtain

$$(\lambda_1\alpha_1+\lambda_2\alpha_2+\lambda_3\alpha_3-\mu)d_\alpha=0,$$

where the d_α are the Taylor coefficients. Since

$$\partial^3 g/\partial z_1\partial z_2\partial z_3\equiv 0 \quad\text{we have}\quad \alpha_1\alpha_2\alpha_3 d_\alpha=0.$$

From these two equalities we obtain that there exists no more than one index α such that $d_\alpha \neq 0$. Thus,

$$g(z) = d_\alpha z^\alpha,$$

but this function belongs to W only if $d_\alpha = 0$.

One can examine analogously the case when $k=4$, the origin is an inner point of the convex hull of the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of A, but is not an inner point of the convex hull of any three of these eigenvalues.

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Спректральный синтез для операторов типа Эйлера

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Пусть $A=(a_{ij})_{i,j=1}^{k}$ — квадратная матрица k-того порядка, $a_{ij}\in \mathbf{C}$. Сопоставим ей следующий дифференциальный оператор:

$$D_A = \sum_{i,j=1}^{k} a_{ij} z_j \frac{\partial}{\partial z_i}.$$

В работе доказано: если матрица A и область $U\subset \mathbf{C}^k$ таковы, что для любой точки $Z\in U$ замыкание множества

$$\{e^{tA}z\colon\ t \leqq 0\}$$

является компактом, лежащим в области U, то любое замкнутое D_A-инвариантное подпространство W пространства $H(U)$ допускает спектральный синтез, то есть корневые функции оператора D_A, попавшие в пространство W, полны в нем.

Приведены примеры, показывающие точность этого результата.

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ОТДЕЛ ФИЗИКИ И МАТЕМАТИКИ БАШКИРСКОГО ФИЛИАЛА АН СССР

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On p-Helson sets in $\mathbf{R}^n$

V. N. DEMENKO

In this work we are going to investigate the structure of p-Helson sets in $\mathbf{R}^n$. A closed subset of $\mathbf{R}^n$ is said to be p-Helson if every continuous function defined on this set can be extended to a function of class $A_p(\mathbf{R}^n)$. (Here $A_p(\mathbf{R}^n)$ stands for the system of functions with p-summable Fourier series.) Historically, the first result concerning sets with this property is due to Helson who has proved that no interpolating set for $A(\mathbf{R})$ can be the support of any measure with Fourier transform tending to zero at infinity. The theory of Helson sets for $p=1$ has been rather completely elaborated up to now (for more details see [4]). It turned out that the arithmetical nature of sets has a great effect on their interpolating properties. Even though not every countable compact set is a Helson set. At the same time examples are known for Helson sets with relatively simple structure in spaces of dimension greater than one. The existence of Helson curves in $\mathbf{R}^2$ and $\mathbf{R}^3$ was proved by KAHANE (cf. [4, Chapter VII. 9]). In [6] MCGEHEE and WOODWARD constructed not only Helson curves in $\mathbf{R}^2$ but even Helson k-manifolds in $\mathbf{R}^{lk}$ ($l \geqq k+1$). This result was generalized by MÜLLER [7] by having given examples for Helson k-manifolds in any space $\mathbf{R}^n$ ($n \geqq k+1$).

The results of the present work relate to the less elucidated case $p>1$. It is not difficult to prove that every countable compact set is p-Helson for $p>1$. It is much more complicated to determine which compact sets of measure zero are p-Helson. So, OLEVSKIĬ constructed a compact set of measure zero which is not p-Helson for any $p<2$ (see [8, Chapter IV]). In [9] OLEVSKIĬ stated a hypothesis on the connection between A_p-interpolating properties of sets and their metric characteristics expressed by the Hausdorff dimension. In this work p-Helson sets will be investigated from this aspect. We shall prove the following statement.

Theorem 1. *Let E be a compact set of Hausdorff dimension $2n/q_0$ in $\mathbf{R}^n$. Then E is p-Helson for every $p>q_0/(q_0-1)$.*

The metric estimate appearing in Theorem 1 is unimprovable, more exactly, the following is true.

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Theorem 2. *For every real number* $q_0 \geqq 2$ *and integer* $n \geqq 1$ *there exists a compact set* $E \subset \mathbf{R}^n$ *of Hausdorff dimension* $2n/q_0$ *such that* E *is not* p*-Helson for any* $p < q_0/(q_0-1)$.

Moreover, the condition of Theorem 1 is not necessary that a set be p-Helson. It was proved by the author in [2] that a polygon with finite break points, whose Hausdorff dimension is 1 in $\mathbf{R}^n$, is a p-Helson set in $\mathbf{R}^2$ for any $p > 1$. The same is true in $\mathbf{R}^n$ as well $(n > 2)$.

As for the simplicity of the structure of p-Helson sets in $\mathbf{R}^n$, Theorem 3 will show that for every preassigned $p_0 \in (1, 2)$ and $n > 1$ a continuous curve can be constructed such that it is p-Helson for every $p > p_0$ but is not p-Helson for $p < p_0$.

In the last part of the paper the relation between the classes $A_p(\mathbf{R}^n)$ and $A_p(T^n)$ will be investigated. (Here T means the interval $(-\pi, \pi)$.) For the case $p=1$ it was shown by Wiener (see [4, p. 20]) that if $\operatorname{supp} f \subset T$ then f belongs to the class $A(\mathbf{R})$ exactly when f belongs to $A(T)$; moreover, the condition concerning the strict inclusion of the support of f in the interval T cannot be weakened. In the case $p > 1$ this inclusion is possibly not strict.

Theorem 4. *Let* $p > 1$, supp $f \subseteqq T^n$, *and assume that* g *is an extension of* f *to* $\mathbf{R}^n$ *such that* g *is* 2π*-periodic with respect to each coordinate. Then* $f \in A_p(\mathbf{R}^n)$ *if and only if* $g \in A_p(T^n)$.

From Theorem 4 it follows that the statements of the first three theorems are true for $A_p(T^n)$ as well.

Now we proceed with the proofs of our statements. We shall use the following notations: $\hat{f}(y)$ stands for the Fourier transform of the function $f(x)$, that is,

$$\hat{f}(y) = \int_{\mathbf{R}^n} f(x) e^{-ixy}\, dx.$$

$M(\mathbf{R}^n)$ denotes the Banach space of Borel measures of bounded variation on $\mathbf{R}^n$ with the norm

$$\|\mu\| = \int_{\mathbf{R}^n} |d\mu|.$$

$M(E)$ is the subspace of $M(\mathbf{R}^n)$ consisting of those measures which are concentrated on E, and $\hat{\mu}(y)$ means the Fourier transform of the measure μ:

$$\hat{\mu}(y) = \int_{\mathbf{R}^n} e^{-ixy}\, d\mu(x).$$

The sequence $\{r_k(t)\}$ $(1 \leqq k \leqq \infty)$ denotes the Rademacher system, that is,

$$r_k(t) = \operatorname{sgn} \sin 2^k \pi t, \quad 0 \leqq t \leqq 1.$$

For a set G, $\chi_G(x)$ means the characteristic function of G, and $|E|$ denotes the Lebesgue measure of the set $E(\subset \mathbf{R}^n)$.

We say that $f \in A_p(\mathbf{R}^n)$ $(f \in A_p(T^n))$ if the Fourier transform of the function f is integrable on the p-th power.

For given $E \subset \mathbf{R}^n$ and $0 < \alpha \leqq n$ we set

$$H_\alpha(E) = \sup_{\varepsilon > 0} \big(\inf \big(\sum_j |\text{diam } V_j|^\alpha\big)\big),$$

where the greatest lower bound is taken with respect to all those systems of neighbourhoods which satisfy the conditions $E \subset \bigcup_j V_j$ and diam $V_j < \varepsilon$. The number

$$\dim E = \sup \big(\alpha \colon H_\alpha(E) > 0\big)$$

is called the Hausdorff dimension of the set E.

In the sequel we set $q = p/(p-1)$ and C will denote constants, possibly different in different occurrences.

The following theorem was proved by Salem (see [5, p. 106]).

Theorem. *If* $2 < q_0 < \infty$ *and* $q > q_0$, *then there exist a closed set* $E \subset T$ *satisfying the condition* $\dim E = 2/q_0$ *and a nonzero positive measure* μ *concentrated onto* E*, such that* $\hat{\mu} \in l^q(Z)$.

We shall also need the following result, which was established by the author in [2, Lemma 1].

Proposition 1. *Assume that there exists a measure* $\mu \in M(E)$ *on the set* $E \subset \mathbf{R}^n$ *such that* $\|\hat{\mu}\|_q < \infty$ $(q > 2)$. *Then* E *is not a* p*-Helson set in* $\mathbf{R}^n$.

In the case $n = 1$ and $q_0 > 2$, Theorem 2 immediately follows from these two statements and from Theorem 4 to be proved later.

In order to solve the analogous problems in $\mathbf{R}^n$ we have to form the direct product of sets:

$$E^{(n)} = E \times \ldots \times E \subset \mathbf{R}^n,$$

and the corresponding direct product of measures:

$$\mu^{(n)} = \mu \otimes \ldots \otimes \mu \subset M(\mathbf{R}^n).$$

Then

$$\operatorname{supp} \mu^{(n)} \subseteqq E^{(n)},$$

$$\|\hat{\mu}^{(n)}\|_{L_q(\mathbf{R}^n)} = \|\hat{\mu}\|^n_{L_q(\mathbf{R})} < \infty, \quad q > q_0,$$

and the set $E^{(n)}$ will not be p-Helson for $p < p_0$. In addition,

$$\dim E^{(n)} = n(\dim E) = 2n/q_0.$$

For $q_0 = 2$ we take the pair: $E = T^n$ and the Lebesgue measure μ on T^n.

To prove Theorem 3 a construction will be described by the aid of which a new proof can be given for Salem's theorem.

Let us introduce the constant

$$\beta_p(E) = \inf \|\chi_G\|_p,$$

where the lower bound is taken with respect to all open sets G containing E.

It was shown in [2, Lemma 2] that if the constant $\beta_{p'}(E)$ is equal to zero for some $p' < p$, then E is a p-Helson set on the plane. A noncomplicated transfer of the proof of this result to the n-dimensional case gives the following

Proposition 2. *Let E be a compact set in $\mathbf{R}^n$ such that for some reals $p \in (1, \infty)$ and $p' \in (1, p)$*

$$\beta_{p'}(E) = 0$$

holds. Then E is a p-Helson set in $\mathbf{R}^n$.

By an elementary cube of rank s in $\mathbf{R}^n$ we shall mean an object of the form

$$\prod_{l=1}^{n} \left[\frac{k^{(l)}}{2^s}, \frac{k^{(l)}+1}{2^s}\right],$$

where every $k^{(l)}$ is an integer $(l=1, \ldots, n)$.

Lemma 1. *Let $\{\Delta_j\}$ be a collection of elementary n-dimensional cubes with disjoint inner parts and let r_s denote the number of cubes of rank s in $\{\Delta_j\}$. Then, for every $p \in (1, 2)$, we have*

$$\|\sum_j \chi_{\Delta_j}\|_p < C \sum_s r_s^{1/2} 2^{-ns/q},$$

where C is a constant depending only on p.

Proof. Let us select the cubes of rank s from $\{\Delta_j\}$, and denote them by δ_j:

$$\delta_j = \prod_{l=1}^{n} \left[\frac{k_j^{(l)}}{2^s}, \frac{k_j^{(l)}+1}{2^s}\right].$$

Let Q_l $(l=(l_1, \ldots, l_n))$ denote the cube in $\mathbf{R}^n$ with edges parallel to the coordinate axes, with sides $2^{s+1}\pi$ and midpoint $x^{(l)}$, where

$$x_j^{(l)} = 2^s \pi (2|l_j| - 1) \operatorname{sgn} l_j, \quad l_j = \pm 1, \pm 2, \ldots.$$

Introducing the function

$$\chi_s(x) = \sum_{j=1}^{r_s} \chi_{\delta_j}(x),$$

we have

$$\|\hat{\chi}_s\|_{L_p(Q_l)} = \Big\|\sum_{j=1}^{r_s} \exp\Big(-i\sum_{m=1}^{n} k_j^{(m)} y_m 2^{-s}\Big) \prod_{m=1}^{n} (\exp(-iy_m 2^{-s})-1)/y_m\Big\|_{L_p(Q_l)} \leqq$$

$$\leqq \Big\|\prod_{m=1}^{n} (\exp(-iy_m 2^{-s})-1)/y_m\Big\|_{C(Q_l)} \Big\|\sum_{j=1}^{r_s} \exp\Big(-i\sum_{m=1}^{n} k_j^{(m)} y_m 2^{-s}\Big)\Big\|_{L_p(Q_l)} \leqq$$

$$\leqq \frac{C}{2^{ns}\prod_{m=1}^{n}|l_m|} \Big\|\sum_{j=1}^{r_s} \exp\Big(-i\sum_{m=1}^{n} k_j^{(m)} y_m\Big)\Big\|_{L_p(T^n)} 2^{ns/p};$$

(1)
$$\|\hat{\chi}_s\|_{L_p(Q_l)} \leqq C r_s^{1/2} 2^{-ns/q} \prod_{m=1}^{n} |l_m|^{-1}.$$

Combining (1) with respect to all Q_l we get

$$\|\hat{\chi}_s\|_p = \Big(\sum_{l} C r_s^{p/2} 2^{-nsp/q} \prod_{m=1}^{n} |l_m|^{-p}\Big)^{1/p} = C r_s^{1/2} 2^{-ns/q}.$$

Then

$$\Big\|\sum_j \|\hat{\chi}_{\Delta_j p} \leqq \sum_s \|\hat{\chi}_s\|_p \leqq C \sum_s r_s^{1/2} 2^{-ns/q},$$

and Lemma 1 is proved.

P r o o f o f T h e o r e m 1. Without restricting generality we can assume that E is included in the unit cube of the space $\mathbf{R}^n$. We first remark that q_0 cannot be less than 2 since the Hausdorff dimension of sets in $\mathbf{R}^n$ does not exceed n. The case $q_0=2$ is also evident since for $p\geqq 2$ every compact set in $\mathbf{R}^n$ is p-Helson. Hence, let $q_0>2$, and let us given arbitrary $p\in(p_0, 2)$, $\varepsilon>0$ and $\alpha\in(2n/q_0, 2n/q)$. From the definition of Hausdorff dimension it follows that there exists a covering $\{V_j\}$ of the set E such that

$$d_j = \operatorname{diam} V_j \leqq 1 \quad (j=1, 2, \ldots), \quad \sum_{j=1}^{\infty} d_j^{\alpha} < \varepsilon.$$

If $k_j=[|\log_2 d_j|]$, then V_j belongs to the union of $m_j(\leqq 2^n)$ elementrary cubes of rank k_j. Let us denote these cubes by Δ_{jl} $(l=1, \ldots, m_j)$, and consider the union

$$D = \bigcup_{j=1}^{\infty}\Big(\bigcup_{l=1}^{m_j} \Delta_{jl}\Big).$$

Let r_s be the number of cubes of rank s in D. We have

(2)
$$E\subset \bigcup_{j=1}^{\infty} V_j \subset D,$$

$$\sum_{s=0}^{\infty} r_s 2^{-\alpha s n} < \sum_{j=1}^{\infty} 2^{n+\alpha} d_j^{\alpha} < 2^{n+\alpha}\varepsilon,$$

whence $r_s < C\varepsilon 2^{\alpha sn}$. Then

$$\sum_{s=0}^{\infty} r_s^{1/2} 2^{-ns/q} < C\varepsilon^{1/2} \sum_{s=0}^{\infty} 2^{\sqrt{s}n(\alpha/2-1/q)} = C\varepsilon^{1/2},$$

and the estimate of Lemma 1 yields that

$$\|\hat{\chi}_D\|_p < C\varepsilon^{1/2}. \tag{3}$$

Taking into account that ε and p were arbitrary points in the corresponding intervals, (2) and (3) mean that $\beta_p(E)=0$ for $p>p_0$. Application of Proposition 2 completes the proof of the theorem.

The following two simple lemmas are of technical feature.

We shall need the Hinchin inequality

$$A'_p\Big(\sum_{j=1}^{m} \alpha_j^2\Big)^{1/2} \leq \Big\|\sum_{j=1}^{m} \alpha_j r_j(t)\Big\|_{L_p[0,1]} \leq A_p\Big(\sum_{j=1}^{m} \alpha_j^2\Big)^{1/2}, \quad 1 \leq p < \infty.$$

Lemma 2. *There exists a constant $M>0$ such that for every real $q\in[1, \infty)$, integer $m>M$, and sequence $\{k_j\}$ $(j=1, \ldots, m)$ of numbers one can find a collection of signs $\{\varepsilon_j\}$ $(j=1, \ldots, m;\ \varepsilon_j=\pm 1)$ such that*

$$\Big|\sum_{j=1}^{m} \varepsilon_j\Big| < 2m^{1/2}, \tag{4}$$

$$\Big\|\sum_{j=1}^{m} \varepsilon_j \exp(-ik_j x)\Big\|_{L_q(T)} < (8\pi)^{1/q} \tau^{-1/q} A_q m^{1/2}, \tag{5}$$

where $\tau = \int_{-2}^{2} \exp(-y^2/2)\,dy$.

Proof. It is known (see [3, p. 196]) that

$$2^{-m} \sum_{|m/2-j|<\sqrt{m}} C_m^j \to \tau \quad \text{as} \quad m \to \infty.$$

Let M be such that

$$2^{-m} \sum_{|m/2-j|<\sqrt{m}} C_m^j > \tau/2 \quad \text{for} \quad m > M. \tag{6}$$

By the Hinchin inequality

$$\int_0^1 \Big|\sum_{j=1}^{m} r_j(t) \exp(-ik_j x)\Big|^q dt < A_q^q m^{q/2} \quad \text{for} \quad x\in T.$$

Integrating this inequality with respect to x from $-\pi$ to π and changing the order of integrations gives

$$\int_0^1 \int_{-\pi}^{\pi} \Big|\sum_{j=1}^{m} r_j(t) \exp(-ik_j x)\Big|^q dx\, dt < A_q^q m^{q/2} 2\pi. \tag{7}$$

Applying the Chebyshev inequality in the outer integral of (7) yields

$$(8)\qquad \left|\left\{t\in[0,1]:\ \int_{-\pi}^{\pi}\Big|\sum_{j=1}^{m} r_j(t)\exp(-ik_jx)\Big|^q\,dx > \frac{8A_q^q m^{q/2}\pi}{\tau}\right\}\right| < \frac{\tau}{4}.$$

Inequality (6) means that for $m>M$ at least the $\tau/2$-th part of the whole collection of signs $\{\varepsilon_j\}$ $(j=1, \dots, m)$ satisfies condition (4), and it follows from (8) that, in addition, only the $\tau/4$-th part of $\{\varepsilon_j\}$ can fail to satisfy (5). Consequently, we can find a collection with the required properties. (Such collections make up at least the $\tau/4$-th part of all collections.)

Lemma 2 is proved.

Lemma 3. *Let the collection* m_j $(j=0, \dots, l)$ *satisfy the conditions*

$$(9)\qquad m_0 > 2^l\cdot 200,\quad m_j/2-m_j^{1/2} < m_{j+1} < m_j/2+m_j^{1/2},\quad j=0, \dots, l-1.$$

Then

$$(10)\qquad m_0 2^{-j}-6m_0^{1/2}2^{-j/2} < m_j < m_0 2^{-j}+6m_0^{1/2}2^{-j/2},\quad j=1, \dots, l.$$

Proof. Assume that (10) is proved for $j=s$ $(s\geqq 0)$. Then

$$m_0 2^{-(s+1)}-3m_0^{1/2}2^{-s/2}-(m_0 2^{-s}+6m_0^{1/2}2^{-s/2})^{1/2} < m_{s+1} <$$
$$< m_0 2^{-(s+1)}+3m_0^{1/2}2^{-s/2}+(m_0 2^{-s}+6m_0^{1/2}2^{-s/2})^{1/2}.$$

By (9) for $s<l$ we have

$$3m_0^{1/2}2^{-s/2}+(m_0 2^{-s}+6m_0^{1/2}2^{-s/2})^{1/2} < 6m_0^{1/2}2^{-(s+1)/2}$$

and Lemma 3 is proved.

It will be convenient to assume that $M>400$, where M is the constant occurring in Lemma 2.

Lemma 4. *Let us given an elementary segment Δ and a real number $q>2$. Then for any integers N and l satisfying*

$$(11)\qquad 2^N|\Delta| > 2^{l+1}M,$$

the segment Δ can be splitted into 2^l parts T_j $(j=0, \dots, 2^l-1)$, which are the unions of elementary segments of rank N and such that

(12)

$$\left\|\chi_\Delta-\chi_{T_j}\frac{|\Delta|}{|T_j|}\right\|_q = O(|\Delta|^{1/2}2^{N(1/q-1/2)+l/2}+|\Delta|^{-1/2}2^{(l-N)/2}\|\chi_\Delta\|_q),\quad j=0, \dots, 2^l-1.$$

Proof. Without restricting generality we can assume that 0 is the left endpoint of Δ. Let N and l be integers satisfying (11). Applying Lemma 2 to the collection $\Lambda=\{1, 2, \dots, 2^N|\Delta|\}$, let $\{\varepsilon^{(j)}\}$ denote the selected collection of signs. Let us de-

compose Λ into two parts Λ_0 and Λ_1, where Λ_0 is the set of indices corresponding to plus signs, and Λ_1 is the same to minus signs in $\{\varepsilon^{(j)}\}$. Setting $m_\sigma=|\Lambda_\sigma|$ $(\sigma=0,1)$, $m=|\Lambda|$, it follows from Lemma 2 that

$$m/2-m^{1/2}<m_\sigma<m/2+m^{1/2},\quad \sigma=0,1.$$

Similarly to the first step, by the aid of Lemma 2 we can split every Λ_{σ_1} into two parts $\Lambda_{\sigma_1\sigma_2}$ $(\sigma_1,\sigma_2\in\{0,1\})$, where $\Lambda_{\sigma_1 0}$ is the set of indices corresponding to plus signs in $\{\varepsilon^{(j)}_{\sigma_1}\}$, and $\Lambda_{\sigma_1 1}$ is the same to minus signs in $\{\varepsilon^{(j)}_{\sigma_1}\}$; and so on. In addition, we always have

$$1/2m_{\sigma_1,\dots,\sigma_s}-m^{1/2}_{\sigma_1,\dots,\sigma_s}<m_{\sigma_1,\dots,\sigma_{s+1}}<1/2m_{\sigma_1,\dots,\sigma_s}+m^{1/2}_{\sigma_1,\dots,\sigma_s}\quad (0\leqq s\leqq l-1).$$

Each of the sequences $m, m_{\sigma_1},\dots,m_{\sigma_1\dots\sigma_l}$ satisfies the conditions of Lemma 3, and therefore,

$$m2^{-l}-6m^{1/2}2^{-l/2}<m_{\sigma_1,\dots,\sigma_s}<m2^{-l}+6m^{1/2}2^{-l/2} \tag{13}$$

and

$$M<m_{\sigma_1,\dots,\sigma_s}<m2^{-(s-1)},\quad s=1,\dots,l.$$

Let us fix an arbitrary collection $\sigma_1,\dots,\sigma_l$. In order to simplify indices we write only s in place of $\sigma_1,\dots,\sigma_s$. Let $\Lambda_s=\{k^{(j)}_s\}$ $(1\leqq j\leqq m_s)$, let $\chi^{(j)}_s$ denote the characteristic function of the segment $[k^{(j)}_s 2^{-N},(k^{(j)}_s+1)2^{-N}]$, and set

$$g_s(x)=\sum_{j=1}^{m_s}\chi^{(j)}_s(x),\quad g_0(x)=\chi_\Delta(x).$$

We are going to estimate $\|\hat g_s-2\hat g_{s+1}\|_q$ $(s=0,\dots,l-1)$:

$$\|\hat g_s-2\hat g_{s+1}\|_q^q=\Big\|\sum_{j=1}^{m_s}\varepsilon^{(j)}_s\hat\chi^{(j)}_s\Big\|_q^q=\Big\|(\exp(-iy2^{-N})-1)/y\sum_{j=1}^{m_s}\varepsilon^{(j)}_s\exp(-ik^{(j)}_s y2^{-N})\Big\|_q^q=$$

$$=2^{N(1-q)}\int_{-\infty}^{\infty}\Big|\Big(\sum_{j=1}^{m_s}\varepsilon^{(j)}_s\exp(-ik^{(j)}_s(t))(e^{-it}-1)/t\Big|^q dt\leqq$$

$$\leqq 2^{N(1-q)}\int_T\Big|\sum_{j=1}^{m_s}\varepsilon^{(j)}_s\exp(-ik^{(j)}_s t)\Big|^q dt\,3\Big(\sum_{r=1}^{\infty}r^{-q}\Big)\leqq$$

$$\leqq C2^{N(1-q)}m_s^{q/2}<Cm2^{N(1-q)-(s-1)q/2},$$

that is,

$$\|\hat g_s-2\hat g_{s+1}\|_q<C|\Delta|^{1/2}2^{N(1/q-1/2)-s/2}. \tag{14}$$

Combining the estimates (14) for all $s=0,\dots,l-1$ we get

$$\|2^l\hat g_l-\hat g_0\|_q\leqq\sum_{s=0}^{l-1}\|2^{s+1}\hat g_{s+1}-2^s\hat g_s\|_q=\sum_{s=0}^{l-1}2^s\|2\hat g_{s+1}-\hat g_s\|_q=$$

$$=C|\Delta|^{1/2}2^{N(1/q-1/2)}\sum_{s=0}^{l-1}2^{s/2}=C|\Delta|^{1/2}2^{N(1/q-1/2)+l/2}.$$

Let $\gamma = m/m_l$. On account of (13) we can write that

$$-6(m^{1/2}+6\cdot 2^{l/2})^{-1}2^{l/2} < \gamma 2^{-l}-1 < 6(m^{1/2}-6\cdot 2^{l/2})^{-1}2^{l/2},$$

or

(15) $$|2^{-l}\gamma-1| < 6(m^{1/2}-6\cdot 2^{l/2})^{-1}2^{l/2} < 12|\Delta|^{-1/2}2^{(l-N)/2}.$$

Now we estimate $\|\hat{g}_0-\gamma\hat{g}_l\|_q$:

$$\|\hat{g}_0-\gamma\hat{g}_l\|_q \leqq \|\hat{g}_0-2^l\hat{g}_l\|_q+|1-\gamma 2^{-l}|\|2\hat{g}_l\|_q \leqq$$
$$\leqq C|\Delta|^{1/2}2^{N(1/q-1/2)+l/2}+|1-\gamma 2^{-l}|(\|\hat{g}_0\|_q+C|\Delta|^{1/2}2^{N(1/q-1/2)+l/2}).$$

By (15) and (11) we obtain

$$\|\hat{g}_0-\gamma\hat{g}_l\|_q < C(|\Delta|^{1/2}2^{N(1/q-1/2)+l/2}+|\Delta|^{-1/2}2^{(l-N)/2}\|\hat{g}_0\|_q).$$

Setting

$$T_j = \sum_{k\in\Lambda_{\sigma_1,\dots,\sigma_l}} \Delta_k,$$

where

$$\Delta_k = [k2^{-N},(k+1)2^{-N}],\quad j=\sum_{s=1}^{l}\sigma_s 2^{s-1},\quad j=0,\dots,2^l-1,$$

Lemma 4 is proved.

Corollary to Lemma 4. *Let us be given an elementary segment Δ, real numbers $q>2$, $\varepsilon>0$, and an integer l. Then for some N the segment Δ can be splitted into disjoint subsets T_j, which are elementary segments of rank N, such that*

$$\|\hat{\chi}_\Delta-\hat{\chi}_{T_j}|\Delta|/|T_j|\|_q < \varepsilon,\quad j=0,\dots,2^l-1.$$

Lemma 5. *Let Δ be an elementary segment, and let $q>2$, $\varepsilon>0$, and $\alpha>2/q$ be constants. Then for every sufficiently large number N there exists a collection $\{\tau_j\}$ $(j=1,\dots,m)$ of elementary segments of rank N such that*

(17) $$m2^{-\alpha N} < |\Delta|,$$

(18) $$\Big\|\hat{\chi}_\Delta-h\sum_{j=1}^{m}\hat{\chi}_{\tau_j}\Big\|_q < \varepsilon,$$

where

(19) $$h = |\Delta|2^N m^{-1}.$$

Proof. Applying Lemma 4 with $|\Delta|2^{\alpha N}>8M$ and $l=[N(1-\alpha)]+1$, we can take the collection $\{\tau_j\}$ of segments of rank N, constituting any set T_s occurring in Lemma 4.

Lemma 6. *For every $k=0,1,\dots$, let $E_k\subset\mathbf{R}^n$ be a compact set and let $\mu_k\in M(\mathbf{R}^n)$ be a measure satisfying for some $q>2$ the following conditions:*

$$\text{(I) } E_{k+1}\subset E_k,\quad \text{(II) } \mu_k\in M(E_k),\quad \text{(III) } E_k=\bigcup_{j=1}^{m_k} e_j^{(k)},$$

the sets $e_j^{(k)}$ *are compact and* $|e_j^{(k)} \cap e_l^{(k)}| = 0$ *if* $j \neq l$,

(IV) $$\mu_0(E_0) = 1; \quad \mu_{k+1}(e_j^{(k)}) = \mu_k(e_j^{(k)}) > 0, \quad j = 1, \dots, m_k,$$

(V) $$\operatorname{diam} e_j^{(k)} < 2^{-k}, \quad j = 1, \dots, m_k,$$

(VI) $$\|\hat{\mu}_k\|_q < C.$$

Then the sequence $\{\mu_k\}$ *weakly converges to a measure* μ *satisfying*

(20) $$\mu \in M(E), \quad E = \bigcap_{k=0}^{\infty} E_k,$$

(21) $$\|\hat{\mu}\|_q < C.$$

Proof. First we prove that $\{\mu_k\}$ is weakly convergent. Let $f(x)$ be a continuous function on $\mathbf{R}^n$, and $\omega(f, \delta)$ denote the modulus of continuity of the restriction of f onto E_0. Let $x_j^{(k)}$ be any representative of $e_j^{(k)}$ $(x_j^{(k)} \in e_j^{(k)})$. By (IV) and (V) it follows that if $l > k$ then

$$\Big|\int_{\mathbf{R}^n} f\,d\mu_k - \int_{\mathbf{R}^n} f\,d\mu_l\Big| = \Big|\sum_{j=1}^{m_k}\Big(\int_{e_j^{(k)}} f\,d\mu_k - \int_{e_j^{(k)}} f\,d\mu_l\Big)\Big| \leqq$$

$$\leqq \sum_{j=1}^{m_k}\Big|\int_{e_j^{(k)}} (f(x) - f(x_j^{(k)}))\,d\mu_k - \int_{e_j^{(k)}} (f(x) - f(x_j^{(k)}))\,d\mu_l\Big| \leqq$$

$$\leqq \sum_{j=1}^{m_k} 2\omega(f, 2^{-k})\mu_k(e_j^{(k)}) = 2\omega(f, 2^{-k}).$$

Since E_0 is bounded and closed, hence

$$\lim_{k, l \to \infty}\Big|\int_{R^n} f\,d\mu_k - \int_{R^n} f\,d\mu_l\Big| = 0.$$

Let us take now a continuous function f such that

$$\varrho(\operatorname{supp} f, E) = \delta > 0.$$

By (IV) we infer that E_k is contained in the 2^{-k}-neighbourhood of the set E, and therefore, $\operatorname{supp} f \cap E_k = \emptyset$ holds for $k > |\log_2 \delta|$. Consequently,

$$\int_{R^n} f\,d\mu = \lim_{k \to \infty} \int_{R^n} f\,d\mu_k = 0, \quad (\operatorname{supp} f) \cap E = \emptyset,$$

which gives (20).

Let us examine the sequence of Fourier transforms of the measures μ_k at the point $y \in \mathbf{R}^n$:

(22) $$|\hat{\mu}_k(y) - \hat{\mu}(y)| < 2\omega(e^{-ixy}, 2^{-k}) < |y| 2^{1-k}.$$

By (22) it can be seen that $\hat{\mu}_k(y)$ tends to $\hat{\mu}(y)$ everywhere on $\mathbf{R}^n$. Applying Fatou's theorem to $|\hat{\mu}_k(y)|^q$ we obtain (21). The proof of Lemma 6 is completed.

Let $\Delta=[\alpha, \beta]$. For any $(0<\Theta<1)$, $\Theta\Delta$ denotes the segment

$$\Theta\Delta = [\alpha, \alpha+\Theta(\beta-\alpha)],$$

which is the compression of Δ by Θ with respect to the left endpoint. Simple calculation results in the following:

$$\|\hat{\chi}_\Delta-\hat{\chi}_{\theta\Delta}/\theta\|_q = \left\|(\exp(-i|\Delta|y)-1)/iy-(\exp(-i\theta|\Delta|y)-1)/\theta iy\right\|_q =$$
$$= \left\|((1-\theta)(1-\exp(-i|\Delta|y)))-\exp(-i\theta|\Delta|y)(1-\exp(-i(1-\theta)|\Delta|y))/\theta y\right\|_q <$$
$$< C(p)|\Delta|^{1/p}(1-\theta+(1-\theta)^{1/p})/\theta.$$

We shall use this estimate for $\Theta \in (1/2, 1)$ in the form

$$\left\|\hat{\chi}_\Delta-\frac{1}{\theta}\hat{\chi}_{\theta\Delta}\right\|_q < B_p|\Delta|^{1/p}(1-\theta)^{1/p}. \tag{25}$$

In what follows $\mathbf{x}=(x_1, \ldots, x_n)$ denotes a point of $\mathbf{R}^n$.

Theorem 3. *For every real $q_0\geqq 2$ and integer $n\geqq 1$ there exists a continuous mapping from* [0, 1] *into $\mathbf{R}^n$ whose graph is a p-Helson set in $\mathbf{R}^n$ for every $p>p_0$, but which is not p-Helson for $p<p_0$.*

Proof. We construct sequences of sets E_k and measures μ_k satisfying the conditions of Lemma 6 for any $q>q_0$ and such that

(a) E_k is composed of elementary cubes $e_j^{(k)}$ with sides δ_k and with point $x_j^{(k)}$ closest to the zero $(j=1, \ldots, m_k)$;

(b) the measure μ_k is given by the density function

$$p_k(\mathbf{x}) = \sum_{j=1}^{m_k} h_j^{(k)}\chi_{e_j^{(k)}}(\mathbf{x});$$

(c) $$m_{2l+1}\delta_{2l+1}^{n\alpha} \to 0 \quad \text{as} \quad l \to\infty, \quad \alpha = 2/q_0.$$

Let $P_r(G)$ denote the projection of the set G onto the axis $0x_1$. The sets E have to fulfil the relations

(d) $$\varrho\big(P_r(E_{2l}\cap e_r^{(2l-1)}),\ P_r(E_{2l}\cap e_s^{(2l-1)})\big) \geqq \delta_{2l},$$
$$r, s = 1, \ldots, m_{2l-1}; \quad l = 1, 2, \ldots; \quad r \neq s.$$

Set

$$\delta_0 = 1, \quad \mathbf{x}_1^{(0)} = (0, \ldots, 0), \quad m_0 = 1, \quad h_1^{(0)} = 1.$$

Assume that E_j and μ_j have been already constructed for every $0\leqq j\leqq k$, where k is even. We take $q_k=q_0+2^{-k}$ and $\Delta=[0, \delta_k]$. Let $\|\hat{\chi}_\Delta\|_{q_k}$ be denoted by ξ and choose a real $\varepsilon>0$ such that

$$\Big(\sum_{j=1}^{m_k} h_j^{(k)}\Big)((\xi+\varepsilon)^n-\xi^n) < 2^{-k}.$$

Applying Lemma 5 we fix a collection of elementary segments τ_j of rank N $(j=1, \ldots, s)$ satisfying the conditions

(27) $$\delta_{k+1}^{\alpha} s < \delta_k, \quad \delta_{k+1} = 2^{-N};$$

(28) $$\|\hat{\chi}_\Delta(y) - \hat{\varrho}(y)\|_{q_k} < \varepsilon,$$

where

$$\varrho(x) = |\Delta|(s\delta_{k+1})^{-1} \sum_{j=1}^{s} \chi_{\tau_j}(x).$$

Introduce the following objects

$$\Psi(\mathbf{x}) = \prod_{\nu=1}^{n} \varrho(x_\nu),$$

$$p_{k+1}(\mathbf{x}) = \sum_{j=1}^{m_{k+1}} h_j^{(k+1)} \chi_{e_j^{(k+1)}}(\mathbf{x}) = \sum_{j=1}^{m_{k+1}} h_j^{(k)} \Psi(\mathbf{x} - \mathbf{x}_j^{(k)}),$$

where $e_j^{(k+1)}$ are cubes of rank N and $E_{k+1} = \operatorname{supp} p_{k+1}(x)$.

It is clear from the construction that conditions (I)—(IV) hold, and (V) is also true because of $s>2$. In addition,

(29) $$m_{k+1}\delta_{k+1}^{\alpha n} = m_k s^n \delta_{k+1}^{\alpha n} < m_k \delta_k^n,$$

$$\|\hat{p}_{k+1} - \hat{p}_k\|_{q_k} = \Big\| \sum_{j=1}^{m_k} h_j^{(k)} \exp(-i\mathbf{x}_j^{(k)}\mathbf{y}) \prod_{\nu=1}^{n} \hat{\varrho}(y_\nu) - \sum_{j=1}^{m_k} h_j^{(k)} \exp(-i\mathbf{x}_j^{(k)}\mathbf{y}) \prod_{\nu=1}^{n} \hat{\chi}_\Delta(y_\nu) \Big\|_q \leqq$$

$$\leqq \Big(\sum_{j=1}^{m_k} h_j^{(k)}\Big) \Big\| \prod_{\nu=1}^{n} \hat{\varrho}(y_\nu) - \prod_{\nu=1}^{n} \hat{\chi}_\Delta(y_\nu) \Big\|_{q_k} \leqq \Big(\sum_{j=1}^{m_k} h_j^{(k)}\Big)((\xi+\varepsilon)^n - \xi^n) < 2^{-k}.$$

Now we describe the construction of E_{k+2} and μ_{k+2}. Let $\Delta = [0, \delta_{k+1}]$, $q_{k+1} = q_0 + 2^{-(k+1)}$, $l = [\log_2 m_{k+1}] + 1$, $\xi = \|\hat{\chi}_\Delta\|_{q_{k+1}}$, and let a real $\varepsilon > 0$ satisfy the condition

(30) $$\Big(\sum_{j=1}^{m_{k+1}} h_j^{(k+1)}\Big)((\xi+2\varepsilon)^n - \varepsilon^n) < 2^{-(k+2)}.$$

By Corollary to Lemma 4 the segment Δ can be splitted into 2^l disjoint subsets T_j $(j=1, \ldots, 2^l)$ such that

$$T_j = \bigcup_{\nu=1}^{s_j} \tau_\nu^{(j)},$$

where $\tau_\nu^{(j)}$ is an elementary segment of rank N and

$$\|\hat{\chi}_\Delta - \alpha_j \hat{\chi}_{T_j}\|_{q_{k+1}} < \varepsilon,$$

where $\alpha_j = |\Delta| 2^N / s_j$ is a normalizing coefficient.

Let $\Theta = 1 - 2^{-r}$, where r is an integer satisfying

$$B_p 2^{-(r+N)/p_{k+1}} < \varepsilon\big(\max_{1 \leqq j \leqq 2^l} \alpha_j s_j\big)^{-1}.$$

Then by (25) we get that

$$\left\| \hat{\chi}_\tau - \frac{1}{\theta} \hat{\chi}_{\theta\tau} \right\|_{q_{k+1}} < \varepsilon \left(\max_{1 \leqq j \leqq 2^l} \alpha_j s_j \right)^{-1}$$

holds for every elementary segment τ of rank N. Let ΘT_j stand for the set

$$\theta T_j \leqq \bigcup_{\nu=1}^{s_j} \theta \tau_\nu^{(j)}.$$

Let us introduce the functions

$$\psi_j(x) = \frac{\alpha_j}{\theta} \chi_{\theta T_j}(x), \quad \Psi_j(\mathbf{x}) = \prod_{\nu=1}^{n} \psi_j(x_\nu),$$

and set

$$p_{k+2}(\mathbf{x}) = \sum_{j=1}^{m_{k+2}} h_j^{(k+2)} \chi_{e_j^{(k+2)}}(\mathbf{x}) = \sum_{j=1}^{m_{k+1}} h_j^{(k+1)} \Psi_j(\mathbf{x} - \mathbf{x}_j^{(k+1)}),$$

$e^{(k+2)}$ being a cube of rank Nr, $E_{k+2} = \operatorname{supp} p_{k+2}(x)$, $\delta_{k+2} = 2^{-Nr}$, and $x_j^{(k+2)}$ is the point closest to zero in $e^{(k+2)}$. Condition (d) is implied by the fact that the distance between the supports of any two different functions $\psi_j(x)$ is not less than 2^{-Nr}. It can be also seen from the construction that

(32)
$$m_{k+2} \delta_{k+2}^n < m_{k+2} \delta_{k+1}^n,$$

that is by (29),

$$m_{k+1} \ \delta_{k+1}^{\alpha n} < m_{k-1} \delta_{k-1}^n$$

(k is positive and even), whence (c) follows. Further, we have

$$\|\hat{\psi}_j - \hat{\chi}_\Delta\|_{q_{k+1}} \leqq \|\alpha_j \hat{\chi}_{T_j} - \hat{\chi}_\Delta\|_{q_{k+1}} + \left\| \alpha_j \hat{\chi}_{T_j} - \frac{\alpha_j}{\theta} \hat{\chi}_{\theta T_j} \right\|_{q_{k+1}} \leqq$$

$$\leqq \varepsilon + \left(\max_{1 \leqq j \leqq 2^l} \alpha_j s_j \right) \left\| \hat{\chi}_\tau - \frac{1}{\theta} \hat{\chi}_{\theta\tau} \right\|_{q_{k+1}} < 2\varepsilon,$$

where τ is an elementary segment of rank N. By the last inequality and (30) we obtain

$$\|\hat{p}_{k+2} - \hat{p}_{k+1}\|_{q_{k+1}} = \left\| \sum_{j=1}^{m_{k+1}} h_j^{(k+1)} \exp(-i\mathbf{x}_j^{(k+1)}\mathbf{y}) \left(\prod_{\nu=1}^{n} \hat{\psi}_j(y_\nu) - \prod_{\nu=1}^{n} \hat{\chi}_\Delta(y_\nu) \right) \right\|_{q_{k+1}} \leqq$$

$$\leqq \left(\sum_{j=1}^{m_{k+1}} h_j^{(k+1)} \right) \left\| \prod_{\nu=1}^{n} \hat{\psi}_j(y_\nu) - \prod_{\nu=1}^{n} \chi_\Delta(y_\nu) \right\|_{q_{k+1}} < \left(\sum_{j=1}^{m_{k+1}} h_j^{(k+1)} \right) ((\xi + 2\varepsilon)^n - \xi^n) < 2^{-(k+2)}.$$

Thus, conditions (I)—(V) of Lemma 6 have been verified.

Now we estimate the norm $\|\hat{\mu}_k\|_q$ for $q>q_0$. Take a number k_0 such that $q_{k_0}<q$, and let $k>k_0$. Since $\|\hat{\mu}_k\|=1$, we infer that $\|\hat{\mu}_k-\hat{\mu}_{k+1}\|_C\leqq 2$. Then

$$\|\hat{\mu}_k-\hat{\mu}_{k+1}\|_q \leqq \|\hat{\mu}_k-\hat{\mu}_{k+1}\|_{q_k}^{q_k/q}2^{1-q_k/q} < 2^{1-kq_0/q},$$

$$\|\hat{\mu}_k\|_q < \|\hat{\mu}_{k_0}\|_q+\sum_{s=k_0}^{k-1}\|\hat{\mu}_s-\hat{\mu}_{s+1}\|_q < \|\hat{\mu}_{k_0}\|_q+\sum_{s=k_0}^{\infty}2^{1-sq_0/q} = C(q).$$

Moreover, $\hat{\mu}_{k_0}\in L_q(\mathbf{R}^n)$ for any $q>1$. So, condition (VI) is also proved.

Thus, the sequence of measures μ_k weakly converges to a measure μ such that

$$\operatorname{supp}\mu \subseteqq E = \bigcap_k E_k, \quad \|\hat{\mu}\|_q < C(q)$$

for any $q>q_0$. This means that the set E together with any set containing E is not p-Helson for $p<p_0$. At the same time, it follows from (c) that for $q<q_0$ and odd k we have

$$m_k^{1/2}d_k^{n/q} = (m_k d_k^{n\alpha})^{1/2}d_k^{(1/q-1/q_0)n} \to 0, \quad \text{as} \quad k\to\infty.$$

Consequently, by Lemma 1 we infer that for every $p>p_0$

$$\liminf_{k\to\infty}\|\hat{\chi}_{E_k}\|_p = 0;$$

hence

(34) $$\beta_p(E) = 0.$$

We remark that (c) and (V) also imply that $\dim E\leqq 2n/q_0$; whence, by Theorem 1, it follows that $\dim E=2n/q_0$.

Let us consider the set $F=Pr(E)$. By (d) we infer that for any two points $\mathbf{x}', \mathbf{x}''\in F$, the inequality $\varrho(\mathbf{x}', \mathbf{x}'')<d_{2l}$ implies that the inverse images belong to the same cube $e_j^{(2l-1)}$. Since $d_{2l}\to 0$ and the set E is closed, the mapping $\mathbf{x}\to Pr(\mathbf{x})$ is one-to-one and continuous between F and E. Let the inverse mapping Pr^{-1}: $F\to E$ be denoted by f. The function f defined on the closed set $F\subset[0, 1]$ is continuous. We extend $f(x)$ in a standard way to a continuous function given on the whole interval [0, 1] by linear interpolation on the intervals adjacent to F. The graph of the function

$$f\colon [0, 1] \to \mathbf{R}^{n-1}$$

can be obtained from F by supplementing countably many disjoint intervals. It was shown in [2] that $\beta_p(\tau)=0$ $(p>1)$ for any finite interval $\tau\in\mathbf{R}^2$. The same is true for any one-dimensional interval τ in $\mathbf{R}^n$ for $n>2$, as well. To see this it is enough to take δ-neighbourhoods V_δ of the interval τ in $\mathbf{R}^n$. Then

(35) $$\|\hat{\chi}_{V_\delta}\|_p = O(\delta^{(n-1)/q}),$$

and by (34), (35) it follows that $\beta_p(\Gamma_f)=0$ $(p>p_0)$.

We note that for $q_0<2n$ the Hausdorff dimension of the curve we have constructed will be the same as the one of the set E, that is $2n/q_0$. The proof of Theorem 3 is complete.

Finally, we turn to the proof of Theorem 4. It is well-known (see, for example, [1]) that the Hilbert transform

$$\mathscr{H}f(x)=\int_R \frac{f(y)}{x-y}\,dy$$

(the integral is understood in Cauchy's sense of principal value) is a bounded operator from $L_p(\mathbf{R})$ into itself $(1<p<\infty)$. Let $\mathscr{U}_v$ denote the operator of multiplication by the function $\exp(-ivx)$ $(v, x\in\mathbf{R})$:

$$\mathscr{U}_v f(x)=\exp(-ivx)f(x).$$

We shall need the operator $\mathscr{L}$ of convolution with the function

$$\psi(y)=\hat{\chi}_T(y)=2\sin\pi y/y.$$

The operator $\mathscr{L}$ can be expressed in terms of the operators $\mathscr{H}$ and $\mathscr{U}_v$ in the following way:

$$\mathscr{L}=i(\mathscr{U}_\pi\mathscr{H}\mathscr{U}_{-\pi}-\mathscr{U}_{-\pi}\mathscr{H}\mathscr{U}_\pi);$$

whence it can be seen that $\mathscr{L}$ is also a bounded operator acting from $L_p(\mathbf{R})$ into $L_p(\mathbf{R})$. We show that the operators $\mathscr{U}_v$ act from $A_p(T^n)$ into $A_p(T^n)$.

Lemma 7. *For any $p\in(1,\infty)$ there exists a constant α_p such that for every $n\geqq 1$*

(36) $$\sup_{v\in R^n}\|\mathscr{U}_v f\|_{A_p(T^n)}<\alpha_p^n\|f\|_{A_p(T^n)}.$$

Proof. First let $n=1$. We have that

$$(\widehat{\mathscr{U}_v f}(k))=(\hat{f}(k))*(\psi(v+k)).$$

Let us decompose the function ψ into the sum $\psi(y)=\psi_1(y)+\psi_2(y)$, where

$$\psi_1(y)=\begin{cases}0, & y\in[-1,1),\\ 2\sin\pi y/[y], & y\notin[-1,1).\end{cases}$$

Then

(37) $$(\widehat{\mathscr{U}_v f}(k))=(\hat{f}(k))*(\psi_1(v+k))+(\hat{f}(k))*(\psi_2(v+k)).$$

Fix v and set

$$(a_k^{(j)})=(\hat{f}(k))*(\psi_j(v+k)),\quad j=1,2.$$

Consider $(a_k^{(2)})$. The absolute value of $\psi_2(y)$ can be estimated as

$$|\psi_2(y)|<4/(y^2+1).$$

Therefore,

$$(38)\qquad \Big(\sum_k |a_k^{(2)}|^p\Big)^{1/p} \leqq \Big(\sum_k |\hat{f}(k)|^p\Big)^{1/p}\left(\sum_k \frac{4}{(v+k)^2+1}\right) < 20\pi\|f\|_{A_p(T)}.$$

Now we estimate $\|(a_k^{(1)})\|_p^k$. Observe that the l_p-norm of the sequence $(a_k^{(1)})$ does not depend on the integral part of v. It will be convenient to assume that $[v]=0$. We can write

$$(39)\quad a_k^{(1)} = \sum_{m\neq k,\,k+1} 2\hat{f}(m)\frac{\sin\pi(v+k-m)}{k-m} = (-1)^k 2\sin\pi v \sum_{m\neq k,\,k+1} (-1)^m \frac{\hat{f}(m)}{k-m}.$$

Let A_k denote the value

$$A_k = 2\sum_{m\neq k,\,k+1} \frac{(-1)^m\hat{f}(m)}{k-m}.$$

By (39) it follows that

$$(40)\qquad \Big(\sum_k |a_k^{(1)}|^p\Big)^{1/p} = |\sin\pi v|\Big(\sum_k |A_k|^p\Big)^{1/p}.$$

Consider the function $\varrho(y)\in L_p(\mathbf{R})$:

$$\varrho(y) = \sum_k \hat{f}(k)\chi_{[-1/8,\,1/8]}(y-k).$$

Since the operators of convolution by ψ and the summable functions ψ_2 act from $L_p(\mathbf{R})$ into $L_p(\mathbf{R})$, by (37) we infer that

$$\|\psi_1 * \varrho\|_p < C\|\varrho\|_p$$

holds for some constant C depending only on p. On the other hand, for $y\in[k+3/8,\ k+5/8]$ $(k\in\mathbf{Z})$

$$(\psi_1*\varrho)(y) = \int_{-\infty}^{\infty} \psi_1(y-x)\varrho(x)\,dx = \sum_m \hat{f}(m)\int_{m-1/8}^{m+1/8} \psi_1(y-x)\,dx =$$

$$= \sum_{m\neq k,\,k+1} \hat{f}(m)\int_{m-1/8}^{m+1/8} \frac{2\sin\pi(y-x)}{k-m}\,dx = A_k\int_{-1/8}^{1/8} \sin\pi(y-x)\,dx,$$

moreover,

$$\left|\int_{-1/8}^{1/8} \sin\pi(y-x)\,dx\right| > 1/\pi\sqrt{2}.$$

Defining E by

$$E = \bigcup_k\,[k+3/8,\ k+5/8],$$

we have

$$C^p\|\varrho\|_p^p > \|\psi_1*\varrho\|_p^p > \int_E |(\psi_1*\varrho)(y)|^p\,dy \geqq \Big(\sum_k |A_k|^p\Big)\pi^{-p}2^{-(p/2+2)}.$$

Hence we have proved that

(41) $$(\sum_k |A_k|^p)^{1/p} < C\pi 2^{1/2+2/p}\|f\|_{A_p(T)}.$$

Combining (37), (38), (40) and (41) we get

(42) $$\|\mathcal{U}_v f\|_{A_p(T)} \leqq \alpha_p \|f\|_{A_p(T)}.$$

Now we turn to the multidimensional case. Let us introduce the following collection of auxiliary operators $U_{\mathbf{v}}^{(j)}$ $(j=1, \dots, n)$:

$$\mathcal{U}_{\mathbf{v}}^{(j)} f(\mathbf{x}) = \exp(-iv_j x_j) f(\mathbf{x}).$$

Then

(43) $$\mathcal{U}_{\mathbf{v}} f = \mathcal{U}_{\mathbf{v}}^{(1)} \mathcal{U}_{\mathbf{v}}^{(2)} \dots \mathcal{U}_{\mathbf{v}}^{(n)} f.$$

Let us fix an arbitrary number j $(1 \leqq j \leqq n)$. Let J_k $(\mathbf{k} \in \mathbf{Z}^n)$ denote the set

$$J_{\mathbf{k}} = \{\mathbf{m} \in \mathbf{Z}^n \colon m_s = k_s,\ s \neq j\}.$$

We have

$$\widehat{\mathcal{U}_{\mathbf{v}}^{(j)} f}(\mathbf{k}) = \sum_{\mathbf{m} \in J_{\mathbf{k}}} \hat{f}(\mathbf{m}) \psi(k_j + v_j - m_j),$$

and by (42) it follows that

(44) $$\|\mathcal{U}_{\mathbf{v}}^{(j)} f\|_{A_p(T^n)} < \alpha_p \|f\|_{A_p(T^n)}.$$

Combining (43) and (44) gives the required inequality.

Lemma 7 is proved.

Corollary to Lemma 7. *Let $f \in A_p(T^n)$. Then*

$$\|f\chi_{T^n}\|_{A_p(R^n)} \leqq \alpha_p^n \|f\|_{A_p(T^n)}.$$

Proof. We shall use the notation

$$I = [0, 1], \quad g(x) = f(x)\chi_{T^n}(x).$$

Fix $y \in \mathbf{R}^n$ and consider the decomposition $\mathbf{y} = \mathbf{k} + \mathbf{v}$, where $\mathbf{k} \in \mathbf{Z}^n$ and $[v_j] = 0$ $(j=1, \dots, n)$. We have

$$\hat{g}(\mathbf{y}) = \int_{T^n} f(\mathbf{x}) e^{-i\mathbf{x}\mathbf{v}} e^{-i\mathbf{k}\mathbf{x}}\, d\mathbf{x} = \widehat{\mathcal{U}_{\mathbf{v}} f}(\mathbf{k}).$$

Then

$$\|\hat{g}(\mathbf{y})\|_p^p = \sum_{\mathbf{k} \in \mathbf{Z}^n} \int_{I^n} |\widehat{\mathcal{U}_{\mathbf{v}} f}(\mathbf{k})|^p\, d\mathbf{v} = \int_{I^n} \Big(\sum_{\mathbf{k} \in \mathbf{Z}^n} |\widehat{\mathcal{U}_{\mathbf{v}} f}(\mathbf{k})|^p \Big) d\mathbf{v} < \alpha_p^{np} \|f\|_{A_p(T^n)}^p.$$

The following argument approximately repeats the course of the proof of Wiener's theorem in [4, p. 20].

Proof of Theorem 4. Let us take an arbitrary function $f\in A_p(T^n)$. On account of Lemma 7 we get

$$f\chi_{T^n}\in A_p(\mathbf{R}^n).$$

Assume now that $\operatorname{supp} f\subseteqq T^n$, $f\in A_p(\mathbf{R}^n)$, and verify that the extension of f, 2π-periodic in each coordinate, belongs to $A_p(T^n)$. In fact,

$$\sum_{\mathbf{k}\in Z^n}\int_{I^n}|\hat f(\mathbf{y}+\mathbf{k})|^p\,d\mathbf{y}=\int_{R^n}|\hat f(\mathbf{y})|^p\,d\mathbf{y}<\infty;$$

application of Beppo Levi's theorem implies the existence of a point $\mathbf{y}_0$ such that

$$\sum_{\mathbf{k}\in Z^n}|\hat f(\mathbf{y}_0+\mathbf{k})|^p<\infty.$$

As we have

$$\hat f(\mathbf{y}_0+\mathbf{k})=\int_{T^n}f(\mathbf{x})e^{-i\mathbf{y}_0\mathbf{x}}e^{-i\mathbf{k}\mathbf{x}}\,d\mathbf{x},$$

the 2π-periodic extension of the function $g(\mathbf{x})=f(\mathbf{x})\exp(-i\mathbf{y}_0\mathbf{x})$ belongs to $A_p(T^n)$. Since

$$f=\mathscr{U}_{-\mathbf{y}_0}g,$$

we obtain by Lemma 7 that

$$\|f\|_{A_p(T^n)}<\alpha_p^n\|g\|_{A_p(T^n)}<\infty.$$

Theorem 4 is proved.

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О p-хелсоновских множествах в R^n

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В статье изучаются метрические характеристики p-хелсоновских множеств. Замкнутое множество E из $\mathbf{R}^n$ называется p-хелсоновским, если любая функция $f \in C(E)$ может быть непрерывно продолжена до функции класса $A_p(\mathbf{R}^n)$.

Показано, что если хаусдорфова размерность компакта $E \subset \mathbf{R}^n$ есть $2nq_0$, то E — p-хелсоновское множество для всякого $p > q_0(q_0 - 1)$. Этот результат не может быть улучшен: для любых $q_0 > 2$ и $n \geqq 1$ существует компакт $E \subset \mathbf{R}^n$ хаусдорфовой размерности $2nq_0$, не являющийся p-хелсоновским множеством при $p < p_0$.

Доказано также, что если $\operatorname{supp} f \subseteqq [-\pi, \pi]^n$, то функция f принадлежит классу $A_p(\mathbf{R}^n)$ тогда и только тогда, когда она принадлежит классу $A_p([-\pi, \pi]^n)$ $(1 < p < \infty)$.

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A test of convergence of Fourier series with respect to multiplicative systems, analogous to the Jordan test

V. I. SHCHERBAKOV

Introduction

I. A group of sequences G and the variation of functions on it. Let $p_0=1$, $\{p_n\}_{n=1}^{\infty}$ be a sequence of integers such that $p_n\geqq 2$, $m_n=\prod_{k=1}^{n} p_k$ $(n=0, 1, 2, \ldots)$,

$$G = \{\{x_n\}_{n=1}^{\infty}:\ x_n = 0, 1, \ldots, p_n-1\}$$

a group with the operation $\{x_n\}_{n=1}^{\infty}+\{y_n\}_{n=1}^{\infty}=\{(x_n+y_n) \bmod p_n\}_{n=1}^{\infty}$ and $\dot{-}$ its inverse operation. The map $G\to[0, 1]$ defined by

$$G\ni x = \{x_n\}_{n=1}^{\infty} \to |x| = \sum_{n=1}^{\infty} (x_n/m_n)\in[0, 1] \tag{1}$$

is bijective everywhere except the points

$$|x| = \frac{l}{m_n}\in]0, 1[;\quad l = 1, 2, \ldots, m_n-1;\quad n = 1, 2, \ldots. \tag{2}$$

Points of the form (2) have two preimages in G, one of which is finite, i.e., all but finitely many elements of the sequence are 0. The finite preimage of $\frac{l}{m_n}$ in G will be denoted by $\frac{l}{m_n}+$, and the infinite one by $\frac{l}{m_n}-$.

We also introduce the notations:

$$G_n = \left[0, \frac{1}{m_n}-\right] = \left\{x\in G:\ 0 \leqq x \leqq \frac{1}{m_n}-\right\} =$$

$$= \{x = x_k\}_{k=1}^{n}:\ x_k = 0 \quad\text{for}\quad k = 1, 2, \ldots, n\},\ G_{l,n} = \frac{l}{m_n}+G_n = \left[\frac{l}{m_n}, \frac{l+1}{m_n}-\right];$$

here $C(G)$ is the set of continuous (with respect to the topology of the group G (see e.g. [1, p. 18])) functions, where by functions we shall mean maps of G into the set $\mathbf{C}$ of complex numbers.

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It is clear that

(3) $$x \dot{-} G_n = x \dot{+} G_n = \tilde{x}_n \dot{+} G_n = \left[\tilde{x}_n, \left(\tilde{x}_n + \frac{1}{m_n}\right)\right],$$

where $\tilde{x}_n \in G$ is a $\{p_n\}$-rational point (i.e., all but finitely many x_k are 0) such that

(4) $$|\tilde{x}_n| = \sum_{k=1}^{n} \frac{x_k}{m_k}$$

($|x|$ is defined by formula (1)).

II. Multiplicative system of Price. Consider the following system of functions

$$\psi_0(x) \equiv 1;$$

$$\psi_m(x) = \exp\left(\frac{2\pi il}{p_{k+1}}\right) \quad \text{if} \quad x \in G_{l,k+1}; \quad l = 0, 1, \dots, m_{k+1}-1; \quad k = 0, 1, \dots;$$

$$\psi_n(x) = \prod_{k=0}^{s} (\psi_{m_k}(x))^{a_k} \quad \text{if} \quad n = \sum_{k=0}^{s} a_k m_k;$$

where a_k and s are integers, $0 \leqq a_k < p_{k+1}$, and $a_s \neq 0$.

This is a complete orthonormal system and the partial sums of the corresponding Fourier series (see [8, 3. II.]) are

$$S_n(x, f) = \int_G f(x \dot{-} t) D_n(t)\, dt = \int_G f(t) D_n(x \dot{-} t)\, dt = \int_G f(t) \overline{D_n(t \dot{-} x)}\, dt,$$

where $D_n(t) = \sum_{k=0}^{n-1} \psi_k(t)$ is the Dirichlet kernel.

The following equalities are well-known (see [4, Lemma 3] and [8, 2.2]):

(5) $$D_n(x) = \frac{1-(\psi_{m_s}(x))^{a_s}}{1-\psi_{m_s}(x)} D_{m_s}(x) + (\psi_{m_s}(x))^{a_s} D_{n'}(x) \quad \text{for any} \quad x \in G,$$

where $n = a_s m_s + n'$; a_s and n' are integers, $1 \leqq a_s < p_{s+1}$ and $0 \leqq n' < m_s$;

(6) $$D_{m_k}(x) = m_k \quad \text{if} \quad x \in G_k, \quad \text{and} \quad D_{m_k}(x) = 0 \quad \text{for} \quad x \in G - G_k.$$

The following two statements are known (see e.g. [7. Lemma 1, 2]).

Lemma 1. *If* $n = \sum_{j=k}^{s} a_j m_j + n^{(k)}$, *where* a_j, $n^{(k)}$, *and* s *are integers such that* $0 \leqq a_j < p_{j+1}$, $a_s \neq 0$ *and* $0 \leqq n^{(k)} < m_k$, *then for every* $x \in G - G_k$ *we have*

$$D_n(x) = D_{n^{(k)}}(x) \psi_{n-n^{(k)}}(x).$$

Lemma 2. *If* $n \geqq m_k$ *and* $l = 1, 2, \dots, m_k - 1$, *then*

$$\int_{G_{l,k}} D_n(x)\, dx = 0 \quad \textit{and} \quad \int_{G-G_k} D_n(x)\, dx = \sum_{l=1}^{m_k-1} \int_{G_{l,k}} D_n(x)\, dx = 0.$$

Lemma 3. *For any integers k and n the following inequality holds:*

$$0 \leqq \int_{G_k - G_{k+1}} D_n(t)\,dt \leqq 1.$$

In particular, the value of this integral is real.

Proof. If $n \geqq m_{k+1}$, then from Lemma 2 we have

$$\int_{G_k - G_{k+1}} D_n(t)\,dt = \sum_{l=1}^{p_{k+1}-1} \int_{G_{l,k+1}} D_n(t)\,dt = 0.$$

If $n < m_{k+1}$, then $D_n(t) = n$ for $t \in G_{k+1}$, since $\psi_j(t) = 1$ when $j < n < m_{k+1}$, and $t \in G_{k+1}$ and $D_n(t) = \sum_{j=0}^{n-1} \psi_j(t)$. So, $\int_{G_{k+1}} D_n(t)\,dt = \frac{n}{m_{k+1}}$ (similarly

$$\int_{G_{l+1}} D_n(t)\,dt = \frac{n}{m_{l+1}} \quad \text{for} \quad n < m_{l+1} \quad (l = 1, 2, \ldots, k)\Big)$$

and then using the equality $\int_G D_n(t)\,dt = 1$ for any integer n (by Lemma 2,

$$\int_{G_k} D_n(t)\,dt = \int_G D_n(t)\,dt - \int_{G-G_k} D_n(t)\,dt = 1 \quad \text{holds if} \quad n \geqq m_k\Big),$$

we have

$$\int_{G_k - G_{k+1}} D_n(t)\,dt = \int_{G_k} D_n(t)\,dt - \int_{G_{k+1}} D_n(t)\,dt = \begin{cases} \dfrac{n}{m_k} - \dfrac{n}{m_{k+1}} & \text{if} \quad n < m_k, \\ 1 - \dfrac{n}{m_{k+1}} & \text{if} \quad m_k \leqq n < m_{k+1}. \end{cases}$$

The case $n < m_k$ can be treated analogously. Lemma 3 is proved.

III. The function $q(t)$ and an estimate of the Dirichlet kernel. Put

$$G_{n,+} = \bigcup_{l=1}^{b_n} G_{l,n+1}, \quad G_{n,-} = \dot{-} G_{n,+} = \bigcup_{l=a_n}^{p_{n+1}-1} G_{l,n+1},$$

where

$$b_n = \left[\frac{p_{n+1}}{2}\right], \quad a_n = \left[\frac{p_{n+1}+1}{2}\right], \tag{7}$$

and $[y]$ means the integral part of $y \in \mathbf{R}$.

It is clear that, for $(n = 1, 2, \ldots)$, $G_{n,+} \cup G_{n,-} = G_n - G_{n+1}$ and

$$G_n \cap G_{n+1} = \begin{cases} \emptyset & \text{if} \quad p \quad \text{is odd}, \\ G_{p_{\frac{n+1}{2}}, n+1} & \text{if} \quad p_{n+1} \quad \text{is even}. \end{cases}$$

On $G-\{0\}$ we define the function

$$q(x) = \frac{m_n}{\sin \frac{\pi l}{p_{n+1}}} \quad \text{if} \quad x \in G_{l,n+1}, \quad l = 1, 2, \ldots, p_{n+1}-1; \quad n = 0, 1, 2, \ldots.$$

In [7. p. 135] the following relations are proved:

$$(8) \qquad c_1 \ln p_{n+1} \leqq \int_{G_n - G_{n+1}} q(t)\, dt \leqq c_2 \ln p_{n+1} \quad (n = 0, 1, 2, \ldots),$$

where $c_1 > 0$ and c_2 are constants and

$$(9) \qquad m_n \leqq q(x) \leqq m_{n+1}/2 \quad \text{if} \quad x \in G_n - G_{n+1}.$$

In [6, Theorem 1] the following estimate is obtained: for every integer $n \geqq 1$ and $x \in G-\{0\}$

$$(10) \qquad |D_n(x)| \leqq 2q(x).$$

A simple consequence of (9) and (10) is the following (proved in [8, 3.6.]): for every integer $n \geqq 1$, $k \geqq 1$, and $t \in G-G_k$,

$$(11) \qquad |D_n(t)| \leqq m_k.$$

§ 1. Some lemmas on the variation of functions

Let $P_k = \{x = \{x_k\}_{n=1}^{\infty} \in G \ : \ x_k = 0 \text{ or } x_k = p_k - 1\}$ (the coordinates of $\{x_k\}_{n=1}^{\infty} \in P_k$ other than the k-th one are arbitrary). Then

$$(12) \quad P_k = \left[0, \frac{1}{m_k} - \right] \cup \left[1 - \frac{1}{m_k}, 1\right] \cup \left(\bigcup_{l=1}^{m_k - 1} \left[\frac{l}{m_{k-1}} - \frac{1}{m_k}; \left(\frac{l}{m_{k-1}} + \frac{l}{m_k} \right) - \right] \right).$$

From equation (12) it follows easily that

$$(13) \qquad \mu(P_k) = 2/p_k,$$

where $\mu(E)$ denotes the measure of the measurable set $E \subset G$.

Put $P = \limsup_{k\to\infty} P_k = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} P_k$. (13) implies that

$$(14) \qquad \mu(P) \leqq 2/\limsup_{n\to\infty} p_n,$$

and if $\limsup_{n\to\infty} p_n = \infty$, then $\mu(P) = 0$. P is the set of all points $\{x_n\}_{n=1}^{\infty} \in G$ for which the coordinates x_k are 0 or $p_k = 1$ except finitely many of them. Set

$$(15) \qquad Q = G - P.$$

Then for every point $x=\{x_n\}_{n=1}^{\infty}\in Q$ one can find an increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, depending on $x\in Q$, such that $x_{n_k}\neq 0$ and $x_{n_k}\neq p_{n_k}-1$. It follows from (14) and (15) that

$$\mu(Q) \geqq 1-\frac{2}{\limsup\limits_{n\to\infty} p_n}, \tag{16}$$

and if $\limsup\limits_{n\to\infty} p_n=\infty$, then $\mu(Q)=\mu(G)=1$.

Note that $P=G$ and $Q=\emptyset$ in the case $p_n\equiv 2$.

Just as in the case of the interval $[a,b]\subset[0,1]$ (see e.g. [3, p. 202]) we define the variation of the function $f(x)$ on the set $E\subset G$ in the following way:

$$V(E)=V(E,f)=\sup_{\substack{y_1,\dots,y_n\in E\\ y_1<y_2<\dots<y_n}}\sum_{k=1}^{n-1}|f(y_{k+1})-f(y_k)|. \tag{17}$$

Let $V(G)$ be the set of functions of bounded variation on G and $CV(G)= =C(G)\cap V(G)$. We have

Lemma 4. *For any point* $x\in Q$, *the inequality*

$$\lim_{n\to\infty} V\big(x\dotplus(G_n-G_{n+1}),f\big)\geqq \Big|\lim_{\substack{t\to x\\ t>x}} f(t)-\lim_{\substack{t\to x\\ t<x}} f(t)\Big| \tag{18}$$

is satisfied under the condition that all quantities in (18) *exist.*

Proof. Set $l_1=\lim\limits_{\substack{t\to x\\ t>x}} f(t)$ and $l_2=\lim\limits_{\substack{t\to x\\ t<x}} f(t)$.

Let $x=\{x_n\}_{n=1}^{\infty}$. Since $x\in Q$, there exists a sequence of integers $\{n_k\}_{k=1}^{\infty}$ such that $1\leqq x_{n_k}\leqq p_{n_k}-2$ because $x_{n_k}\neq 0$ and $x_{n_k}\neq p_{n_k}$. So,

$$\tilde{x}_{n_k-1}<\tilde{x}_{n_k}<x<(\tilde{x}_{n_k}+1/m_{n_k})-<(\tilde{x}_{n_k-1}+1/m_{n_k-1})-, \tag{19}$$

where $\tilde{x}_{n_k}$ is determined by formula (4).

By (3) we have

$$\tilde{x}_{n_k}-\in x\dotplus(G_{n_k-1}-G_{n_k}) \quad\text{and}\quad \tilde{x}_{n_k}+\frac{1}{m_{n_k}}\in x\dotplus(G_{n_k-1}-G_{n_k}). \tag{20}$$

Let $\varepsilon>0$, then there exists N such that for any integer $k\geqq N$ the following inequalities hold (see (19)):

$$|f(\tilde{x}_{n_k})-l_2|<\varepsilon \quad\text{and}\quad \left|f\left(\tilde{x}_{n_k}+\frac{1}{m_{n_k}}\right)-l_1\right|<\varepsilon. \tag{21}$$

From (20) and (21) we get

$$V(x\dotplus(G_{n_k-1}-G_{n_k}))\geqq\left|f\left(\tilde{x}_{n_k}+\frac{1}{m_{n_k}}\right)-f(\tilde{x}_{n_k}-)-l_1+(l_1-l_2)+l_2\right|\geqq$$

$$\geqq|l_1-l_2|-\left|f\left(\tilde{x}_{n_k}+\frac{1}{m_{n_k}}\right)-l_1\right|-|f(\tilde{x}_{n_k}-)-l_2|>|l_1-l_2|-2\varepsilon.$$

Lemma 4 is proved.

Consequently, if at a point $x\in Q$ we have

$$V(x\dotplus(G_n-G_{n+1}),f)\to 0 \quad\text{as}\quad n\to\infty,\quad\text{then}\quad l=\lim_{t\to x}f(t)$$

exists, so that x is either a point of continuity of $f(t)$ or $f(t)$ has a removable discontinuity in x.

Since

$$\lim_{\substack{t\to l/m_n\\ t<l/m_n}} f(t) \quad\text{and}\quad \lim_{\substack{t\to l/m_n-\\ t>l/m_n-}} f(t)$$

are not defined, the following is true.

Lemma 5. *Every discontinuity of the first kind of the function $f(t)$ is removable at any $\{p_n\}$-rational point $x\in G$ (that is, at a point of the form* (2) *and also at $x=0$ and $x=1$).*

§ 2. Main theorems and their consequences

I. Jordan test for pointwise and uniform convergence. Let $n\geqq 1$ be an integer and $f(t)\in V(G)$.

Theorem 1. 1) *If $f(t)$ is continuous at a point $x\in G$, then the following inequality is true:*

$$|S_n(x;f)-f(x)|\leqq 2c_2V(x\dotplus(G_k-G_{k+1}),f)\ln p_{k+1}+o(1),\quad n\to\infty, \tag{22}$$

where the constant c_2 is determined by (8) *and the integers n and k satisfy*

$$m_k\leqq n<m_{k+1}. \tag{23}$$

2) *If $f(t)\in CV(G)$, then* (22) *holds uniformly on G.*

The following statement follows easily from Theorem 1.